# ON THE LOGARITHMS OF
# NEGATIVE AND IMAGINARY NUMBERS

*Leonhard Euler*

**1.** In the exchange of letters between Messrs. Leibnitz and Jean Bernoulli, we find a great controversy over the logarithms of negative and imaginary numbers, a controversy which has been treated by both sides with much force, without however, these two illustrious men having fallen into agreement on this matter, although otherwise we note a very perfect harmony between them on all the other points of analysis. This dissension seems all the more remarkable because it centers upon an article of the part of mathematics which we call pure, and which we do not ordinarily believe susceptible to any dispute, it all being based on the most rigorous proofs. For we know that the other questions on which mathematicians are not in agreement belong to the applied part of mathematics, where the diverse ways of envisaging the objects and reducing them to mathematical ideas give rise to real controversies; and we even boast that they have been altogether banished from analysis or pure mathematics.

**2.** Indeed, the glory of infallibility which this science has acquired would suffer a great attack, if there were questions on which beliefs were not only divided, but where it were even impossible to discover the truth by a clear proof which would put an end to all dispute. Since there is not the slightest doubt that such an accommodation between the differing opinions of Messrs. Leibnitz and Bernoulli has not taken place, I will examine both, weighing the arguments which each invokes both for the confirmation of his belief as well as for the refutation of the contrary, and I hope to develop this matter well and put it into full light, so that there will not remain the slightest doubt and so that both parties will be obliged to recognize the solidness of the verdict I will give, and which will put an end to all the disputes which might yet arise on this matter.

**3.** M. Leibnitz gave the first occasion to this controversy with M. Bernoulli, when he advanced in letter CXC that the ratio of 1 to −1, or of −1 to 1, was imaginary, since the logarithm or measure of this ratio is imaginary, where he

clearly was assuming that the logarithms of negative numbers are imaginary or impossible. To this M. Bernoulli declared, in letter CXCIII, that he was not of the same opinion and that he even believed the logarithms of negative numbers were not only real, but also equal to the logarithms of the same numbers taken positively. He holds, then, that

$$\ell(+x) = \ell(-x),$$

which he claims to prove by the equality of their differentials, considering that

$$d\,\ell\,x = \frac{dx}{x}$$

and furthermore

$$d\,\ell(-x) = \frac{-dx}{-x} = \frac{dx}{x}.$$

Against this argument M. Leibnitz responds that the common rule of differentiating logarithms, by dividing the differential of the number by the number itself, is appropriate only for positive numbers and that, consequently, this differentiation $d\,\ell\,x = \frac{dx}{x}$ is appropriate only when $x$ indicates a positive quantity. But I admit that this response, if it were right, would shake the foundation of all of analysis, which consists principally in the generality of the rules and operations which are judged to be true, whatever the nature we suppose the quantities to be that we apply them to.

**4.** But, even though the rule to differentiate logarithms is generally true, so that $d\,\ell\,x = \frac{dx}{x}$ whether $x$ is a positive or negative quantity, the argument of M. Bernoulli does not prove what it claims to prove. I say that just because the differentials of the quantities $\ell(x)$ and $\ell(-x)$ are equal, it does not at all follow that the quantities themselves are equal to each other, because we know that two quantities which differ by a constant have the same differential. Thus the differentials of $x + 1$ and of $x - 1$ are both equal to $dx$, without us being able to conclude an equality between the quantities $x + 1$ and $x - 1$. Besides that, M. Bernoulli could have proved by the same reasoning that $\ell\,2x = \ell\,x$, since

$$d\,\ell\,2x = \frac{2\,dx}{2x} = \frac{dx}{x} = d\,\ell\,x$$

and in general it would follow that $\ell(nx) = \ell(x)$, where $n$ indicates an arbitrary number. That is why from the equality between the differentials of $\ell(x)$ and $\ell(-x)$, we cannot conclude anything, except that these two logarithms differ by a constant quantity. And indeed, $\ell(-x)$, because $-x = x \cdot (-1)$,

is none other than $\ell(x) + \ell(-1)$. It is true that M. Bernoulli claimed that $\ell(-1) = \ell(1) = 0$, in which case we would without doubt have $\ell(-x) = \ell(+x)$, but this is precisely what M. Leibnitz had denied and M. Bernoulli wished to prove by this argument, so that in this area, nothing is yet decided.

**5.** In the same passage which I just examined, M. Bernoulli makes use of still another argument, but one which differs from the preceding only in the way it is said, when he holds that the logarithmic curve has, from two sides of its asymptote, two equal and similar parts, so that to each abscissa or logarithm there correspond two ordinates or equal numbers, one positive and the other negative. For, considering the equation of this curve

$$y\,dx = a\,dy,$$

where $x$ indicates the abscissa taken on the asymptote of the logarithm, $y$ the ordinate, and $a$ the constant subtangent, it would seem to follow that, if to the same abscissa $x$ there corresponds the value $y = u$, it will correspond also to $y = -u$, since, if $+u\,dx = +a\,du$, we will also have $-u\,dx = -a\,du$. But this reasoning is similar to the preceding, and it would follow from it as well that to each abscissa $x$, there would also correspond the ordinate $y = 2u$, and in general $y = nu$, so that this curve would have not only two ordinates, $y = u$ and $y = -u$, which would correspond to the abscissa $x$, but the number of ordinates would be infinite. This is a consequence, however, that we are quite a long way from admitting. From this we see that this argument does not prove that the logarithm has two similar branches on two sides of its asymptote.

**6.** But one will perhaps object that it is nevertheless the surest means to judge the shape of a curve and the number of its branches by its equation, and that it is by this principle that the geometers determine the forms of all the algebraic curves. To which I respond that this method has a place only when the equation for the curve is algebraic or at least conceived in finite terms, and that a differential equation is never appropriate to this design. For we know that a differential equation is always indeterminate, because of an arbitrary constant it includes which we must introduce in the integration, so that such an equation always embraces an infinity of curves at the same time. We only have to look at the differential equation for the parabola

$$2y\,dy = a\,dx,$$

and we will see that it contains not only this finite equation

$$y^2 = ax,$$

but also this one

$$y^2 = ax \pm ab,$$

whatever value we give to the quantity $b$. Consequently, by considering only the differential equation $2y\,dy = a\,dx$, we would have to conclude that to the same abscissa $x$ there corresponds not only the ordinate $y = \sqrt{ax}$, but again $y = \sqrt{ax \pm a^2}$, and in general $y = \sqrt{ax \pm ab}$. This reflection is sufficient to show that we cannot judge the shape of a curve by looking at only its differential equation.

**7.** Now, M. Bernoulli, as well as several mathematicians who also hold the same belief, try to prove by still other arguments that the asymptote of the logarithm is at the same time its diameter. These arguments are based either on the construction of this curve, or else on analogy. One makes use of analogy by considering, instead of the equation for the logarithmic $dx = \frac{dy}{y}$, this more general one

$$dx = \frac{dy}{y^n}$$

whose integral is

$$x = C - \frac{1}{(n-1)y^{n-1}}.$$

We note here that whenever $n$ is an odd number, the curve has, without question, a diameter, or two equal and similar branches. This noted, the argument is that we suppose $n = 1$, and since 1 is an odd number, the logarithmic must have the same property. This is, in my opinion, the strongest argument that we have heretofore brought for proving that the logarithmic has a diameter. But I will nevertheless show that this conclusion we want to draw is not sufficiently certain.

**8.** When it is a question, in analysis, of the case of integrability or, in geometry, of certain properties of curves, we rarely find propositions that are general enough, and it is nearly always necessary to except one or several cases which do not apply. One can indeed say that this formula $x^n dx$ is in general integrable, whatever number we put for $n$, provided that we exclude

the case $n = -1$. And it is the same as well with several other general formulas, which we can almost never affirm are integrable in all cases without exception. In the same way, when we say that the equation

$$dx = \frac{dy}{y^n}$$

always represents an algebraic curve, whatever rational number we put for $n$, this proposition would suffer but a single exception, that of the case $n = 1$. Therefore, since this case is specially distinguished from all the others, what will guarantee that it is not necessary to make an exception to the rule mentioned regarding a diameter which we would like to attribute to the curve comprised under the equation $dx = \frac{dy}{y}$? For, in all the other cases where $n$ is equal to an odd number, we recognize with evidence the necessity of a diameter, since in these cases the equation is integrable. But in the case $n = 1$ this evidence ceases entirely, due to the impossibility of integration. Consequently, we are obliged at least to admit that the conclusion we want to draw from this argument is not sufficiently certain.

**9.** One will perhaps doubt that judging the property of a diameter is subject to exceptions similar to what we must recognize in integration. But I will show very clearly that, even for the algebraic curves, it is often necessary to admit some exception relating to the property of diameters. Let us consider, for example, this general equation

$$y = \sqrt{ax} + \sqrt[4]{a^3(b+x)},$$

and we will not hesitate to conclude that this curve always has a diameter, since by reducing it to a rationality we reach an equation of degree 8 where all the exponents of the powers of $y$ are even. But, however certain this conclusion seems, we must exclude from it the case where $b = 0$. For then, if we free the equation $y = \sqrt{ax} + \sqrt[4]{a^3 x}$ of irrationality, we will have precisely

$$y^2 - 2y\sqrt{ax} + ax = \sqrt{a^3 x}$$

or

$$y^2 + ax = (2y + a)\sqrt{ax}$$

and therefore, taking the squares again

$$y^4 + 2axy^2 + a^2x^2 = 4axy^2 + 4a^2xy + a^3x$$

or
$$y^4 - 2axy^2 - 4a^2xy + a^2x^2 - a^3x = 0$$

which, because of the $4a^2xy$ term, is lacking a diameter. So, since in the algebraic curves we are obliged to sometimes recognize exceptions, how can we be assured that the case in question does not require them also? And therefore, more is required for the given argument to prove invincibly that the logarithmic has a diameter.

**10.** The same uncertainty is found in the other arguments we derive from the construction of the logarithmic curve by the quadrature of the hyperbola. For, even when we would turn this construction so that it necessarily results in two branches of the logarithmic, we would still have quite strong reasons for doubting that these two branches necessarily belong together and that they constitute only a single continuous line. To prove this, I could relate several examples of constructions by which we obtain two different curves that are not connected together by the link of continuity. For, since we can always encompass two curves, however different they may be, under one equation, by multiplying their equations together, we only have to imagine such a construction which fits this given composite equation, and it will furnish the two given curves, as if they formed only a single curve. Indeed, having described on the same axis the two parabolas

$$v^2 = ax \quad \text{and} \quad u^4 = a^3x,$$

let us construct from them a new curve whose ordinate $y$, corresponding to the same abscissa $x$, will be equal to the sum of the ordinates $v+u$ of the two given parabolas. Now, since each of these ordinates may be taken positively or negatively, we will find, for each abscissa $x$, four ordinates $v + u$, $-v - u$, $v - u$, $-v + u$, and the constructed curve will have a diameter. Nevertheless, the equation

$$y = v + u + \sqrt{ax} + \sqrt[4]{a^3x}$$

shows us that the curve does not have a diameter, as I just noted in the preceding article.

**11.** Furthermore, just as there are constructions from which we derive two different curves, there are also defective constructions which give only one part of a curve. For, let a circle be described whose diameter is $a$, on which taking the abscissa equal to $x$, the ordinate $y$ will be equal to $\sqrt{ax - x^2}$. Let us then extend each ordinate until it becomes equal to the chord $\sqrt{x^2 + y^2} = \sqrt{ax}$,

and this new ordinate, which we name $z = \sqrt{ax}$, will indicate a parabola. But this description of the parabola does not extend beyond the circle, though the parabola itself extends to infinity. This circumstance proves yet again that it is not always certain to judge the true form of a curve, and all the parts it contains, by whatever construction one might give.

**12.** Moreover, the method itself of judging all the parts which belong to the same curve is appropriate only for algebraic curves. For, after having freed the equation which expresses the nature of the curve of all irrationality, we consider the rational equation which results, whether it contains rational factors or not. In the first case, we judge that the curve is composed of as many different curves as there are factors. But if the equation is not resolvable into rational factors, we conclude that all the points which are indicated by this equation belong to the same curve. This is why, when it is a question of transcendental curves, since the equation itself is not algebraic, we cannot even form the question of whether it has rational factors or not, and therefore the judgment of which parts belong to the same curve is no longer appropriate, if these parts are not directly connected together. And therefore, we are not in a position to decide if the logarithmic has two equal branches which both relate to the same asymptote, or not. At least, we must agree that this decision, whatever it may be, is not necessarily connected to the subject in question, namely whether the logarithms of negative numbers are real or imaginary.

**13.** Logarithms are based on a constant number, taken arbitrarily, whose logarithm we suppose to be equal to 1. Let this number be $e$, and let $x$ indicate the logarithm of the number $y$, so that $x = \ell\, y$, and we will then have $y = e^x$. Therefore, the logarithm $x$ of a given number $y$ is nothing other than the exponent of the power of $e$ which is equal to the number $y$. In the common tables, we assume this arbitrary number $e = 10$, and then $x$ will be the logarithm of the number $y$, if $10^x = y$; and in the logarithms that we call hyperbolic and which have the property that, if $\omega$ indicates an infinitely small fraction, the logarithm of the number $1 + \omega$ is equal to $\omega$, the number $e$, whose the logarithm is 1, becomes equal to 2.718281828459. Now, whatever value we give to this number $e$, provided that it is greater than 1, we see from the formula $y = e^x$ that whenever $y$ is a positive number, it is possible to assign to $x$ a real value, so that $e^x$ becomes equal to $y$. But it is also evident that if $y$ is a negative number, we cannot find a real value for $x$ such that the power $e^x$ becomes negative and equal to $y$.

**14.** It is true, however, that if $x$ is a fraction with an even denominator, the power, or rather the root, $e^x$ may be taken positively as well as negatively, so that if the logarithm $x$ is $\frac{1}{2}$, the number $y$ whose logarithm is $\frac{1}{2}$ may equally well be $-\sqrt{e}$ as $\sqrt{e}$. But this ambiguity is encountered only in the case where $x$ is a fraction whose denominator is an even number, and if the logarithm $x$ were 2, it would certainly be false that 2 would be the logarithm of $y = -ee$, since $-ee$ is not at all equal to $ee$; and therefore it is necessary to at least admit that the logarithms of negative numbers are not in general real. But in regard to the ambiguity of the formula $e^x$ in the case where $x$ is a fraction with an even denominator, I do not know if we can allow it in logarithms. For, taking into account the nature and the usage of logarithms, it seems that to each logarithm there may correspond only one number.

**15.** Be that as it may, we will never prove by similar reasoning that the logarithm of $-1$ is equal to that of $+1$ or to zero, since $e^0$ cannot have values other than $+1$. If someone said that $e^0$ can be regarded as $e^{0/2}$ and therefore as $\sqrt{e^0}$ or $\sqrt{1}$, which would be both $-1$ and $+1$, one would be able, by the same reason, to prove that $x^1$, being equal to $x^{2/2}$, would be equal to both $+x$ and $-x$, and furthermore that $a+x$ would be the same thing as $a-x$, and one would be able to hold, by the same argument, that all quantities are equal to each other. But if the logarithm of $-1$ is not equal to 0, it will necessarily be imaginary. And since $-y = -1 \cdot y$, we will have $\ell(-y) = \ell(-1) + \ell(y)$, from which it is clear that when the logarithm of $+y$ is real, the logarithm of $-y$ must absolutely be imaginary.

**16.** The thesis that each logarithm $x$ may correspond to only a single number $y$ will be again confirmed, when we consider the resolution of the formula $e^x$ in this series

$$e^x = 1 + x + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \frac{x^4}{1 \cdot 2 \cdot 3 \cdot 4} + \frac{x^5}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} + \cdots,$$

where $e$ is the number whose hyperbolic logarithm is equal to 1. This series is regarded in analysis as altogether equivalent to the expression $e^x$, and we cannot doubt that its value is determined as soon as we give to $x$ a fixed value, since this series is always convergent, however large the number we substitute for $x$. And for this reason, we are right to hold that since the expression $e^x$ indicates the number whose logarithm is $x$, it never contains the slightest ambiguity, and that its value is always unique and positive, whatever fraction we may take for $x$, so that even when $x$ is a fraction like $\frac{1}{2}$, the expression $e^x$ will always have only a single positive value.

**17.** If one wanted to insist that the formula $e^x$, in the case of $x = \frac{1}{2}$, had a double value, and that $\frac{1}{2}$ were the logarithm of both $-\sqrt{e}$ and $+\sqrt{e}$, it would follow that the logarithms of imaginary numbers are also real. For let us suppose $x = \frac{1}{3}$, and we know that the formula $e^{1/3}$ contains three values

$$\sqrt[3]{e}, \qquad \frac{-1+\sqrt{-3}}{2}\sqrt[3]{e}, \qquad \frac{-1-\sqrt{-3}}{2}\sqrt[3]{e},$$

each of whose logarithms would be equal to $\frac{1}{3}$. Let us suppose $x = \frac{1}{4}$ and the values of $y = e^{1/4}$ will be

$$\sqrt[4]{e}, \quad -\sqrt[4]{e}, \quad +\sqrt{-1}\sqrt[4]{e}, \quad -\sqrt{-1}\sqrt[4]{e}.$$

Then we would have

$$\ell(+\sqrt{-1}\cdot\sqrt[4]{e}) = \ell(-\sqrt{-1}\cdot\sqrt[4]{e}) = \tfrac{1}{4}.$$

But $\ell\sqrt[4]{e} = \frac{1}{4}\ell e = \frac{1}{4}$, from which it would follow that $\ell\sqrt{-1} = 0$. Now, M. Bernoulli made this beautiful discovery that $\dfrac{\ell\sqrt{-1}}{\sqrt{-1}}$ indicates the quadrature of the circle, and since $\sqrt{-1}$ is unquestionably imaginary, it must be that $\ell\sqrt{-1}$ is also, and so nobody will have less right than M. Bernoulli himself to hold henceforth that $\ell\sqrt{-1}$ is equal to 0.

**18.** So having shown that nobody has yet sufficiently proved that the logarithms of negative numbers are real, but rather that the opposite belief, according to which the logarithms of negative numbers are imaginary, is consistent with the truth, I will put forward the difficulties that we encounter on both sides, whether we hold that the logarithms of negative numbers are real, or we take them to be imaginary. These difficulties will seem so strong, and even so replete with contradictions, that we will be at pain to understand how it would be possible to extricate ourselves from this matter and to safeguard the theory of logarithms. I hope, nevertheless, to achieve this goal.

**19.** If we claim, with M. Bernoulli, that the logarithms of negative numbers are the same as those of the positive numbers, or what comes to the same thing, that $\ell(-1) = 0$, we find the following difficulty: either it is true in general that

$$\ell\, a^x = x\, \ell\, a,$$

or it is not. If it is true, we will have

$$\ell(-1)^x = x\,\ell(-1) = 0$$

which we will easily agree with if $x$ is a whole number. But if $x$ is a fraction, we will have $\ell\sqrt{-1} = 0$ as well, and consequently the reduction of M. Bernoulli of the arcs of the circle to imaginary logarithms would be false. This would be absurd, since this discovery is established by the most solid proofs in analysis. But if we deny that $\ell\,a^x = x\,\ell\,a$, we overturn the entire theory of logarithms, for although we would claim to permit the resolution $\ell\,a^x = x\,\ell\,a$ in the case that $x$ were a whole number, it would nevertheless become altogether useless, if $x$ were to indicate a number in general, or an unknown.

**20.** Let us say, then, with M. de Leibnitz that the logarithms of negative numbers are not real, but imaginary, and we will soon notice that we have fallen into the same trouble. For, let $\ell(-1) = p$, so that $p$ is an imaginary number, and we will not be able to deny that $\ell(-1)^n = np$, especially when $n$ is a whole number. So then let $n = 2$, and we will have

$$\ell(-1)^2 = \ell(+1) = 2p.$$

But in the doctrine of logarithms, it is the first principle that $\ell(+1) = 0$. Consequently, we will have $2p = 0$ and therefore $p = 0$, which is contrary to the hypothesis. We will prove as well that

$$\ell\sqrt{-1}, \quad \ell\,\frac{-1\pm\sqrt{-3}}{2}$$

and the logarithms of the higher roots of unity must all be equal to 0, which results in the same contradictions that were encountered in the preceding hypothesis.

**21.** Notice, then, that we encounter quite palpable contradictions, whichever way we turn. I do not doubt that most mathematicians have noticed this, although they have not judged it appropriate to publish their doubts on this matter, for fear of rendering analysis overly suspect, if they were not in a position to save the theory of logarithms. For it would be without doubt an indelible stain in analysis, if the doctrine of logarithms were so replete with contradictions that it were impossible to find a reconciliation. So for a long time these difficulties tormented me, and I was under several illusions concerning this matter, in order to satisfy myself in some manner, without

being obliged to completely overturn the theory of logarithms. I imagined that, just as a quantity always admits two square roots, three cube roots, four biquadratic roots, etc., in the same manner a quantity would be able to have a double half, a triple third, a quadruple fourth, etc. only one of which would be real, the others imaginary. In this way, setting $\ell\, y = x$, I was thinking that

$$\ell\, \sqrt{y} = \tfrac{1}{2}x \quad \text{and} \quad \ell(-\sqrt{y}) = \tfrac{1}{2}x'$$

and that $\tfrac{1}{2}x$ and $\tfrac{1}{2}x'$ would be different, although the double of either would be the same, equal to $x$. In the same manner, for the three cube roots of $y$, it would be

$$\ell\, \sqrt[3]{y} = \tfrac{1}{3}x, \quad \ell\, \tfrac{-1+\sqrt{-3}}{2}\,\sqrt[3]{y} = \tfrac{1}{3}x' \quad \text{and} \quad \ell\, \tfrac{-1-\sqrt{-3}}{2}\,\sqrt[3]{y} = \tfrac{1}{3}x''$$

where $\tfrac{1}{3}x$, $\tfrac{1}{3}x'$, and $\tfrac{1}{3}x''$ would be different numbers, the first, $\tfrac{1}{3}x$, real, and the two others, $\tfrac{1}{3}x'$ and $\tfrac{1}{3}x''$, imaginary, although the triple of each would be equal to $x$. This explanation seemed to me extremely paradoxical indeed, and unsustainable, but still less absurd than the contradictions that I would have been obliged to admit in the theory of logarithms of negative and imaginary numbers.

**22.** Having gotten a good sense of the importance of all these difficulties which are encountered in the doctrine of logarithms, and which even appear to be open contradictions, one will be hard pressed to understand that it is possible to lift all these difficulties without bringing any attack upon the certitude of analysis and the rules on which it is based. However, the truth is too firmly established for it to be subject to any contradiction, and it is only incorrect ways of looking at it that can blind us. Often, it is so difficult to notice the lack of correctness which is found in our ideas and which shows us such great difficulties, that it seems to us altogether impossible to rescue the truth. This is precisely the case in which we find ourselves in relation to the logarithms of negative and imaginary numbers. For, after having carefully weighed all the difficulties that I just detailed, I have found that they come only from our supposition that each number has only a single logarithm. For, if this supposition were true, it would be indeed certain that we could scarcely find the way to extract ourselves from the predicament this matter puts us in. But, as soon as we grant that a number can have several, and even an infinity, of logarithms, then all the difficulties mentioned lose their force and vanish altogether, and we will recognize the most perfect harmony between all the truths.

**23.** I say, then, that although the number whose logarithm we suppose to be 1 is determined, each number nevertheless has an infinity of logarithms, all of which except for a single one, are imaginary, if the number is positive. But if it is negative or imaginary, all its logarithms will be equally imaginary. As a consequence of this, the logarithm of unity will be not only 0, but there will be yet an infinity of imaginary quantities each of which is just as well a substitute for the logarithm of unity as is 0. So let all the logarithms of unity be

$$0, \ \alpha, \ \beta, \ \gamma, \ \delta, \ \epsilon, \ \zeta, \ \eta, \ \theta, \ \cdots$$

and, since the logarithm of the square root is half the logarithm of the power, and $\sqrt{1}$ is both $+1$ and $-1$, the logarithms of the first value, $+1$, will be

$$0, \ \tfrac{1}{2}\beta, \ \tfrac{1}{2}\delta, \ \tfrac{1}{2}\zeta, \ \tfrac{1}{2}\theta, \ \cdots$$

and the logarithms of the second value, $-1$, will be

$$\tfrac{1}{2}\alpha, \ \tfrac{1}{2}\gamma, \ \tfrac{1}{2}\epsilon, \ \tfrac{1}{2}\eta, \ \cdots$$

which are different from the preceding, although their doubles give the logarithms of unity. Furthermore, taking the cube roots, we will have

$$\ell\, 1 = 0, \ \tfrac{1}{3}\gamma, \ \tfrac{1}{3}\zeta, \ \tfrac{1}{3}\iota, \ \cdots$$
$$\ell\, \frac{-1+\sqrt{-3}}{2} = \ \tfrac{1}{3}\alpha, \ \tfrac{1}{3}\delta, \ \tfrac{1}{3}\eta, \ \cdots$$
$$\ell\, \frac{-1-\sqrt{-3}}{2} = \ \tfrac{1}{3}\beta, \ \tfrac{1}{3}\epsilon, \ \tfrac{1}{3}\theta, \ \cdots$$

and this consideration already wipes out most of the difficulties which troubled us above.

**24.** In order to prove this infinite plurality of logarithms which corresponds to each number, we only have to look at the great relation which is found between the logarithms and the arcs of the circle, since we know that the arcs of the circle can be expresses by imaginary logarithms, and conversely, the logarithms by the imaginary arcs of the circle. Therefore, because the sine or cosine corresponds to the numbers and the arcs to the logarithms, just as the same sine corresponds to an infinity of different arcs, thus it follows that the same number must correspond to an infinity of different logarithms. We understand the circle better than the logarithmic curve, and for this reason consideration of the circle will lead us to a more perfect understanding of logarithms than the logarithmic itself. Furthermore, in the

circle we are able to determine all the arcs which correspond to the same sine or cosine, and although these arcs in the passage to logarithms become imaginary, they will not fail, by convincing us of the infinity of logarithms, to give us an understanding of their expressions and the kind of non-reality under which they are comprised, and that is all we can wish for understanding an imaginary quantity.

**25.** Let $\varphi$ be an arbitrary arc of a circle, whose radius I suppose to be equal to 1. Let $x$ be the sine of the arc and $y$ its cosine, so that $y = \sqrt{1 - x^2}$. Then, calling the circumference of this circle $2\pi$, or the arc of $180° = \pi$, it is clear that all the arcs included in this general expression

$$\pm 2n\pi + \varphi$$

will have not only the same sine, $x$, but also the same cosine, $y = \sqrt{1 - x^2}$, provided that $n$ signifies an arbitrary whole number. Now, since

$$d\varphi = \frac{dx}{y} = \frac{dx}{\sqrt{1 - x^2}},$$

Let us suppose

$$x = zi,$$

and we will have

$$d\varphi = \frac{i\,dz}{\sqrt{1 + z^2}}.$$

But we know that

$$\int \frac{dz}{\sqrt{1 + z^2}} = \ell(\sqrt{1 + z^2} + z) + C.$$

Consequently, we will have

$$\varphi = i\,\ell\left(\sqrt{1 - x^2} + \frac{x}{i}\right) + C,$$

where it is clear that the constant $C$ is equal to zero, since by setting $x = 0$ the arc $\varphi$ must vanish as well. Having then

$$\varphi = i\,\ell\left(\sqrt{1 - x^2} - ix\right),$$

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Translator's note: Most occurrences of $\sqrt{-1}$ in the original are represented here using the symbol $i$.

we will have

$$\varphi = \frac{1}{i}\,\ell(\sqrt{1 - x^2} + ix)$$

or

$$\varphi = \frac{1}{i}\,\ell(y + ix).$$

**26.**   This equation we just found, expressing the relation between the arc $\varphi$ and the sine and cosine, will also apply for all the other arcs which have the same sine, $x$, and cosine, $y$. Consequently, we will have

$$\varphi \pm 2n\pi = \frac{1}{i}\,\ell(y + ix)$$

and therefore

$$\ell(y + ix) = i\,(\varphi \pm 2n\pi).$$

From this it is clear that the same number $y + ix$ corresponds to infinitely many logarithms, which are all included in this general formula $i\,(\varphi \pm 2n\pi)$, where in place of $n$ we can put any whole number we wish. Since $x$ is the sine and $y$ is the cosine of the arc $\varphi$, let us set $x = \sin\varphi$ and $y = \cos\varphi$, and we will have this equality

$$\ell(\cos\varphi + i\sin\varphi) = i\,(\varphi \pm 2n\pi).$$

**27.** From this equation, I will examine the principal cases which will furnish quite a clarification of this matter. So first let $\varphi = 0$, and we will have $\cos\varphi = 1$ and $\sin\varphi = 0$, and the equation we found will give us

$$\ell\,1 = \pm 2n\pi i.$$

Then, putting successively for $n$ all the whole numbers, the logarithms of unity will be

$$\ell\,1 = 0, \quad \pm 2\pi i, \quad \pm 4\pi i, \quad \pm 6\pi i, \quad \pm 8\pi i, \quad \cdots,$$

where we see that, though the logarithm of 1 is 0, like everybody knows it to be, there is also an infinity of imaginary expressions, each of which is as much the logarithm of unity as 0 is.

**28.** This sole consideration puts us in a position to give all the logarithms for any positive number that one might propose. For, let $a$ be the number

proposed and $\alpha$ its hyperbolic logarithm, which we find by ordinary methods, and since

$$\ell\, a = \ell\, 1 + \ell\, a = \alpha + \ell\, 1,$$

all the logarithms of the number $a$ will be

$$\ell\, a = \alpha, \quad \alpha \pm 2\pi i, \quad \alpha \pm 4\pi i, \quad \alpha \pm 6\pi i, \quad \alpha \pm 8\pi i, \quad \cdots,$$

all of which except the first, $\alpha$, are imaginary. It is necessary to note that I am speaking here only of the hyperbolic logarithms to which we are led by integration. But since we know that the various kinds of logarithms all observe a constant ratio between them, everything that I just said and everything I will say about the hyperbolic logarithms will easily be applied to the logarithm tables where we put $\ell\, 10 = 1$, or to any other kind of logarithm.

**29.** Now let the given arc $\varphi$ be 180°, or $\varphi = \pi$, and we will have $\sin \varphi = 0$ and $\cos \varphi = -1$. This assumption made, the general equation found will be changed into this form

$$\ell(-1) = (\pi \pm 2n\pi)i = (1 \pm 2n)\pi i,$$

from which we derive the entire infinity of logarithms of the negative number $-1$, for we will have

$$\ell(-1) = \pm\pi i, \quad \pm 3\pi i, \quad \pm 5\pi i, \quad \pm 7\pi i, \quad \cdots,$$

and from this we clearly see that all the logarithms of $-1$ are imaginary, and all different from the logarithms of $+1$. That notwithstanding, the logarithms of $(-1)^2$, which will be

$$\pm 2\pi i, \quad \pm 6\pi i, \quad \pm 10\pi i, \quad \cdots,$$

are visibly contained in the logarithms of $+1$, which suffices to save us from the apparent contradictions which I made mention of above, although it does not conversely follow that the halves of all the logarithms of $+1$ are logarithms of $-1$, which the very nature of the quantities does not permit, since $-1$ is not the sole square root of $+1$.

**30.** We will now be easily assured that all the logarithms of all the negative numbers are imaginary. For let $-a$ be an arbitrary negative number, and $\alpha$ the logarithm of the positive number $a$, found by the ordinary methods,

so that $\ell(+a) = \alpha$, and because $\ell(-a) = \ell\,a + \ell(-1)$, we will have all the logarithms of the negative number $-a$ expressed as

$$\ell(-a) = \alpha \pm \pi i, \quad \alpha \pm 3\pi i, \quad \alpha \pm 5\pi i, \quad \alpha \pm 7\pi i, \quad \cdots,$$

which are all imaginary. By this, then, the question debated between Messrs. Leibnitz and Bernoulli, namely, whether the logarithms of negative numbers are real or imaginary, is decided in favor of the first, who held them to be imaginary, and all the objections that M. Bernoulli has raised against this belief no longer have any hold on this verdict.

**31.** I will go further, and after having determined the logarithms of both positive and negative numbers, I will go on to imaginary numbers. For this purpose, let $\varphi = \frac{1}{2}\pi$, and we will have $\cos\varphi = 0$ and $\sin\varphi = 1$, from which we derive

$$\ell\,i = \left(\tfrac{1}{2}\pi \pm 2n\pi\right)i = \left(\pm 2n + \tfrac{1}{2}\right)\pi i,$$

and therefore all the logarithms of $+i$ will be

$$\ell(+i) = \tfrac{1}{2}\pi i, \quad -\tfrac{3}{2}\pi i, \quad +\tfrac{5}{2}\pi i, \quad -\tfrac{7}{2}\pi i, \quad +\tfrac{9}{2}\pi i, \quad \cdots.$$

But if we put $\varphi = -\frac{1}{2}\pi$, because $\cos\varphi = 0$ and $\sin\varphi = -1$ we will have

$$\ell(-i) = \left(-\tfrac{1}{2}\pi \pm 2n\pi\right)i$$

and consequently

$$\ell(-i) = -\tfrac{1}{2}\pi i, \quad +\tfrac{3}{2}\pi i, \quad -\tfrac{5}{2}\pi i, \quad +\tfrac{7}{2}\pi i, \quad \cdots.$$

From this it is evident that by adding the logarithms of $+i$ and $-i$ together, in order to get the logarithms of the product, $+1$, we obtain the same logarithms that we found for $+1$. And if we subtract the logarithms of $-i$ from the logarithms of $+i$, in order to get logarithms of the quotient $\frac{+i}{-i} = -1$, we will get the logarithms found for $-1$.

**32.** Let, besides that, $\varphi = \frac{1}{3}\pi$, and we will have $\cos\varphi = \frac{1}{2}$ and $\sin\varphi = \frac{\sqrt{3}}{2}$, from which we will derive

$$\ell\,\frac{1+\sqrt{-3}}{2} = \left(\tfrac{1}{3}\pi \pm 2n\pi\right)i,$$

so that the logarithms of $\frac{1+\sqrt{-3}}{2}$ will be

$$\ell\,\frac{1+\sqrt{-3}}{2} = \tfrac{1}{3}\pi i, \quad -\tfrac{5}{3}\pi i, \quad -\tfrac{11}{3}\pi i, \quad -\tfrac{17}{3}\pi i, \quad \cdots,$$
$$+\tfrac{7}{3}\pi i, \quad +\tfrac{13}{3}\pi i, \quad +\tfrac{19}{3}\pi i, \quad \cdots.$$

Let $\varphi = -\tfrac{1}{3}\pi$, and we will have $\cos\varphi = \tfrac{1}{2}$ and $\sin\varphi = -\tfrac{\sqrt{3}}{2}$, from which we will obtain

$$\ell\,\frac{1-\sqrt{-3}}{2} = \left(-\tfrac{1}{3} \pm 2n\right)\pi i,$$

and therefore

$$\ell\,\frac{1-\sqrt{-3}}{2} = -\tfrac{1}{3}\pi i, \quad +\tfrac{5}{3}\pi i, \quad +\tfrac{11}{3}\pi i, \quad +\tfrac{17}{3}\pi i, \quad \cdots,$$
$$-\tfrac{7}{3}\pi i, \quad -\tfrac{13}{3}\pi i, \quad -\tfrac{19}{3}\pi i, \quad \cdots.$$

Let $\varphi = \tfrac{2}{3}\pi$, so that $\cos\varphi = -\tfrac{1}{2}$ and $\sin\varphi = \tfrac{\sqrt{3}}{2}$, and we will have

$$\ell\,\frac{-1+\sqrt{-3}}{2} = \left(\tfrac{2}{3} \pm 2n\right)\pi i,$$

and

$$\ell\,\frac{-1+\sqrt{-3}}{2} = \tfrac{2}{3}\pi i, \quad -\tfrac{4}{3}\pi i, \quad -\tfrac{10}{3}\pi i, \quad -\tfrac{16}{3}\pi i, \quad \cdots,$$
$$+\tfrac{8}{3}\pi i, \quad +\tfrac{14}{3}\pi i, \quad +\tfrac{20}{3}\pi i, \quad \cdots.$$

Finally, putting $\varphi = -\tfrac{2}{3}\pi$, we will have $\cos\varphi = -\tfrac{1}{2}$ and $\sin\varphi = -\tfrac{\sqrt{3}}{2}$, which gives

$$\ell\,\frac{-1-\sqrt{-3}}{2} = \left(-\tfrac{2}{3} \pm 2n\right)\pi i,$$

and

$$\ell\,\frac{-1-\sqrt{-3}}{2} = -\tfrac{2}{3}\pi i, \quad +\tfrac{4}{3}\pi i, \quad +\tfrac{10}{3}\pi i, \quad +\tfrac{16}{3}\pi i, \quad \cdots,$$
$$-\tfrac{8}{3}\pi i, \quad -\tfrac{14}{3}\pi i, \quad -\tfrac{20}{3}\pi i, \quad \cdots.$$

Since we know that

$$\frac{-1+\sqrt{-3}}{2} \quad \text{and} \quad \frac{-1-\sqrt{-3}}{2}$$

are cube roots of $+1$, however we do the tests, we repeatedly find a marvelous agreement with the truth.

**33.** To get the logarithms of powers, we only have to follow the common rule and multiply the logarithm of the root by the exponent of the power. But, since the root has an infinity of logarithms, we can add together as many

different values as the exponent of the power contains units. In this way, since the logarithms of $(-1)$ were found to be

$$\pm\pi i, \quad \pm 3\pi i, \quad \pm 5\pi i, \quad \pm 7\pi i, \quad \cdots,$$

not only the doubles of these logarithms will give the logarithm of the square $(-1)^2$, but also the sums of two arbitrary ones, and by this means we will obtain all these formulas

$$0, \quad \pm 2\pi i, \quad \pm 4\pi i, \quad \pm 6\pi i, \quad \pm 8\pi i, \quad \cdots$$

which are all the logarithms of $+1$. Furthermore, joining by threes the logarithms of

$$\frac{-1+\sqrt{-3}}{2} \quad \text{or} \quad \frac{-1-\sqrt{-3}}{2}$$

we will likewise obtain all the logarithms of $+1$, since each of

$$\left(\frac{-1+\sqrt{-3}}{2}\right)^3$$

and

$$\left(\frac{-1-\sqrt{-3}}{2}\right)^3$$

is equal to unity.

**34.** This facility for finding the logarithms of powers is also confirmed by the general formula of §26, for it has been proved that

$$(\cos\varphi + i\sin\varphi)^\mu = \cos\mu\varphi + i\sin\mu\varphi,$$

so we have

$$\ell(\cos\varphi + i\sin\varphi)^\mu = \ell(\cos\mu\varphi + i\sin\mu\varphi).$$

This formula being similar to the first, we only have to put $\mu\varphi$ in place of $\varphi$ in the logarithm, and since it is permitted to put $\varphi \pm 2m\pi$ for $\varphi$, we will have

$$\ell(\cos\varphi + i\sin\varphi)^\mu = (\mu\varphi \pm 2\mu m\pi \pm 2n\pi)i,$$

and when the exponent is a fraction $\mu/\nu$ we will have

$$\ell(\cos\varphi + i\sin\varphi)^{\mu/\nu} = \frac{1}{\nu}(\mu\varphi \pm 2\mu m\pi \pm 2\nu n\pi)i,$$

the letters $m$ and $n$ indicating arbitrary whole numbers. Consequently, in the case $\varphi = 0$ and $\varphi = \pi$, we will have

$$\ell\, 1^{\mu/\nu} = \frac{1}{\nu}(\pm 2\mu m \pm 2\nu n)\pi i$$

and

$$\ell(-1)^{\mu/\nu} = \frac{1}{\nu}(\mu \pm 2\mu m \pm 2\nu n)\pi i.$$

And having taken account of this situation, all the difficulties which might be encountered in this matter will disappear entirely, and the doctrine of the logarithms will be safeguarded from all attacks.