Extract of a letter from the elder Mr. Euler to Mr. Bernoulli, concerning the Memoir published among those of 1771

Having read with much pleasure your investigations on numbers of the form $10^p \pm 1$, I have the honor of communicating to you the criteria by which one can judge, for each prime number $2p + 1$, which of the two formulas $10^p + 1$ or $10^p - 1$ will be divisible by $2p + 1$.

For this purpose, it is necessary to distinguish the following two cases.

First Case. If $2p + 1 = 4n + 1$, one has only to consider the divisors of the three numbers n, $n - 2$, and $n - 6$, and if among them one finds either both the numbers 2 and 5, or neither of them, that indicates that the formula $10^p - 1$ will be divisible; but if among the said divisors only one of the numbers 2 or 5 is found, then the formula $10^p + 1$ will be divisible. Thus, for the prime number $2p+1 = 53 = 4n+1$, we will have $n = 13$, and our three numbers will be 13, 11, 7, then neither 2 nor 5 is a divisor, and therefore the formula $10^{26} - 1$ will be divisible by 53.

Second Case. If $2p + 1 = 4n - 1$, one must consider these three numbers n, $n + 2$, and $n + 6$, and if among their divisors either both the numbers 2 and 5 are encountered, or neither of them, then the formula $10^p - 1$ will be divisible; but if only one of the numbers 2 and 5 is found to be among them, then the formula $10^p + 1$ will be divisible. For example if $2p + 1 = 59 = 4n - 1$, and therefore $n = 15$, our three numbers are 15, 17, 21, where 5 is among the divisors but not 2, so the formula $10^{29} + 1$ will be divisible by 59.

These rules are based on a principle whose proof is not yet known.

The largest prime number that we know is without doubt $2^{31} - 1 = 2147483647$, which Fermat has already verified to be prime; and I have also proved it; for, since this formula can admit divisors only of the two forms $248n + 1$ and $248n + 63$, I have examined all the prime numbers contained in these two formulas until 46339, none of which was found to be a divisor.

This progression

$$41,\ 43,\ 47,\ 53,\ 61,\ 71,\ 83,\ 97,\ 113,\ 131,\ \text{etc.}$$

whose general term is

$$41 - x + xx,$$

is all the more remarkable because the first forty terms are all prime numbers.

Extrait d'une lettre de M. Euler le père à M. Bernoulli, concernant le Mémoire imprimé parmi ceux de 1771. p. 318, Nouveaux mémoires de l'académie des sciences de Berlin (1774), 35–36. Number 461 in the Eneström index. Translated by Todd Doucet in 2017.