DISCOVERY OF A MOST EXTRAORDINARY LAW OF NUMBERS, RELATING TO THE SUM OF THEIR DIVISORS

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1. Mathematicians have searched so far in vain to discover some order in the progression of prime numbers, and we have reason to believe that it is a mystery which the human mind will never be able to penetrate. To convince ourselves so, we have only to cast our eyes on the tables of prime numbers, which some have taken the trouble to continue beyond 100,000, and we will notice at once that neither rule nor order reigns. This situation is all the more surprising since arithmetic gives us unfailing rules, by means of which we can continue the progression of these numbers as far as we wish, without however leaving us the slightest trace of any order. I believe myself also to be rather far from this goal, but I have just discovered a very strange law among the sums of the divisors of natural numbers, which at first glance would appear as irregular as the progression of the prime numbers, and which even seems to encompass it. This rule, which I am going to expand upon, is in my opinion all the more important because it is the sort of truth we can persuade ourselves of, without giving a perfect proof. Nevertheless, I will put forth such evidence that we might almost be able to imagine it as equivalent to a rigorous proof.

2. The prime numbers are distinguished from other numbers in that they do not admit any divisors other than unity and themselves. So 7 is a prime number, because it is divisible only by unity and itself. The other numbers which have, besides unity and themselves, still other divisors, are called composites, as for example the number 15, which, besides unity and itself, is divisible by 3 and 5. So in general, if the number p is prime, it will be divisible only by 1 and p; but if p is a composite number, it will have, besides 1 and p, still other divisors. And therefore in the prime case, the sum of the divisors will be $1 + p$, and in the other case, it will be greater than $1 + p$. Since the following thoughts will revolve around the sum of the divisors of each number, I will use a certain character to indicate this. The letter $\int$, which one employs

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in infinite analysis to indicate integrals, when put in front of a number, will mean the sum of its divisors. So $\int 12$ will signify the sum of all the divisors of 12, which is $1+2+3+4+6+12=28$, so that $\int 12 = 28$. That fixed, we will see that $\int 60 = 168$ and $\int 100 = 217$. But since unity has no divisor other than itself, we will have $\int 1 = 1$. Since the number 0 is divisible by every number, the value of $\int 0$ will be infinite. However, in what follows I will assign to it, for each instance put forward, a definite value appropriate to my design.

3. Having so established this sign $\int$ to indicate the sum of the divisors of the number in front of which it is placed, it is clear that, if p indicates a prime number, the value of $\int p$ will be $1+p$, except for the case where $p=1$, because then we have $\int 1 = 1$, and not $\int 1 = 1+1$. From this we see that we must exclude unity from the sequence of prime numbers, so that unity, being the start of whole numbers, it is neither prime nor composite. Now, if the number p is not prime, the value of $\int p$ will be greater than $1+p$. In this case, we will easily find the value of $\int p$ by the factors of the number p. For let a, b, c, d, etc. be distinct prime numbers, and we will easily see that

$$\begin{aligned}
\int ab &= 1+a+b+ab = (1+a)(1+b) = \int a \cdot \int b \\
\int abc &= (1+a)(1+b)(1+c) = \int a \cdot \int b \cdot \int c \\
\int abcd &= (1+a)(1+b)(1+c)(1+d) = \int a \cdot \int b \cdot \int c \cdot \int d \\
&\text{etc.}
\end{aligned}$$

For the powers of prime numbers, we need specific rules such as:

$$\int a^2 = 1+a+a^2 = \frac{a^3-1}{a-1}$$

$$\int a^3 = 1+a+a^2+a^3 = \frac{a^4-1}{a-1}$$

and in general

$$\int a^n = \frac{a^{n+1}-1}{a-1}$$

And by means of these, we will fix the sum of the divisors of each number, however it may be composed, which will be clear by the following formulas:

$$\begin{aligned}
\int a^2 b &= \int a^2 \cdot \int b \\
\int a^3 b^2 &= \int a^3 \cdot \int b^2 \\
\int a^3 b^4 c &= \int a^3 \cdot \int b^4 \cdot \int c
\end{aligned}$$

and in general

$$\textstyle\int a^\alpha b^\beta c^\gamma d^\delta e^\epsilon = \int a^\alpha \cdot \int b^\beta \cdot \int c^\gamma \cdot \int d^\delta \cdot \int e^\epsilon$$

Thus, to find the value of $\int 360$, since 360 resolves into the factors $2^3 \cdot 3^2 \cdot 5$, we will have

$$\textstyle\int 360 = \int (2^3 \cdot 3^2 \cdot 5) = \int 2^3 \cdot \int 3^2 \cdot \int 5 = 15 \cdot 13 \cdot 6 = 1170.$$

4. In order to have in view the progression of the sums of the divisors, I will add the following table, which contains the sums of the divisors of the natural numbers from unity to up to 100:

$\int 1 = 1$	$\int 21 = 32$	$\int 41 = 42$	$\int 61 = 62$	$\int 81 = 121$
$\int 2 = 3$	$\int 22 = 36$	$\int 42 = 96$	$\int 62 = 96$	$\int 82 = 126$
$\int 3 = 4$	$\int 23 = 24$	$\int 43 = 44$	$\int 63 = 104$	$\int 83 = 84$
$\int 4 = 7$	$\int 24 = 60$	$\int 44 = 84$	$\int 64 = 127$	$\int 84 = 224$
$\int 5 = 6$	$\int 25 = 31$	$\int 45 = 78$	$\int 65 = 84$	$\int 85 = 108$
$\int 6 = 12$	$\int 26 = 42$	$\int 46 = 72$	$\int 66 = 144$	$\int 86 = 132$
$\int 7 = 8$	$\int 27 = 40$	$\int 47 = 48$	$\int 67 = 68$	$\int 87 = 120$
$\int 8 = 15$	$\int 28 = 56$	$\int 48 = 124$	$\int 28 = 126$	$\int 88 = 180$
$\int 9 = 13$	$\int 29 = 30$	$\int 49 = 57$	$\int 69 = 96$	$\int 89 = 90$
$\int 10 = 18$	$\int 30 = 72$	$\int 50 = 93$	$\int 70 = 144$	$\int 90 = 234$
$\int 11 = 12$	$\int 31 = 32$	$\int 51 = 72$	$\int 71 = 72$	$\int 91 = 112$
$\int 12 = 28$	$\int 32 = 63$	$\int 52 = 98$	$\int 72 = 195$	$\int 92 = 168$
$\int 13 = 14$	$\int 33 = 48$	$\int 53 = 54$	$\int 73 = 74$	$\int 93 = 128$
$\int 14 = 24$	$\int 34 = 54$	$\int 54 = 120$	$\int 74 = 114$	$\int 94 = 144$
$\int 15 = 24$	$\int 35 = 48$	$\int 55 = 72$	$\int 75 = 124$	$\int 95 = 120$
$\int 16 = 31$	$\int 36 = 91$	$\int 56 = 120$	$\int 76 = 140$	$\int 96 = 252$
$\int 17 = 18$	$\int 37 = 38$	$\int 57 = 80$	$\int 77 = 96$	$\int 97 = 98$
$\int 18 = 39$	$\int 38 = 60$	$\int 58 = 90$	$\int 78 = 168$	$\int 98 = 171$
$\int 19 = 20$	$\int 39 = 56$	$\int 59 = 60$	$\int 79 = 80$	$\int 99 = 156$
$\int 20 = 42$	$\int 40 = 90$	$\int 60 = 168$	$\int 80 = 186$	$\int 100 = 217$

I do not doubt that when one looks at the progression of these numbers, one would nearly lose hope of discovering the least order in it, because the irregularity of the sequence of prime numbers is intermixed with it in such a way that it would at first seem impossible to indicate any law in the progression of these numbers without knowing that of the prime numbers. It even seems that there is more strangeness here than in the prime numbers.

5. Nevertheless, I have noticed that this progression follows a quite regular law, and that it is even the kind of progression that the geometers call *recursive*, so that we can always form each term from those preceding it, according to a constant rule. For if $\int n$ denotes an arbitrary term in this irregular progression, and $\int(n-1)$, $\int(n-2)$, $\int(n-3)$, $\int(n-4)$, $\int(n-5)$, etc., the preceding terms, I say that the value of $\int n$ is always formed from the preceding terms by following this formula:

$$\begin{aligned}\int n = &\int(n-1) + \int(n-2) - \int(n-5) - \int(n-7)\\ &+ \int(n-12) + \int(n-15) - \int(n-22) - \int(n-26)\\ &+ \int(n-35) + \int(n-40) - \int(n-51) - \int(n-57)\\ &+ \int(n-70) + \int(n-77) - \int(n-92) - \int(n-100)\\ &+ \text{etc.}\end{aligned}$$

In this formula, we note:

I. In the alternation of the signs $+$ and $-$, each repeats two at a time.

II. The progression of the numbers 1, 2, 5, 7, 12, 15, etc. which must be successively subtracted from the given number n, will become clear as soon as we take their differences:

N.	1,	2,	5,	7,	12,	15,	22,	26,	35,	40,	51,	57,	70,	77,	92,	100 ...
Diff.	1,	3,	2,	5,	3,	7,	4,	9,	5,	11,	6,	13,	7,	15,	8 ...	

because we have all the natural numbers, 1, 2, 3, 4, 5, 6, etc., alternating with the odd numbers 3, 5, 7, 9, 11, etc., so we can continue the sequence of these numbers as far as we wish.

III. Although this series goes to infinity, we only have to take, in each case, the terms starting where the number after the $\int$ sign is still positive, omitting those that contain negative numbers.

IV. If it happens that the term $\int 0$ appears in this formula, since its value is indeterminate in itself, we must, in each case, instead of $\int 0$ put the given number itself.

6. These things noted, it will not be difficult to apply this formula to any given number and to convince ourselves of its truth, by as many examples as we would wish to develop. And because I must admit that I am not in a position to give a rigorous proof of this law, I will make its correctness seen by a large enough number of examples.

$$\int 1 = \int 0 = 1$$
$$\int 2 = \int 1 + \int 0 = 1 + 2 = 3$$
$$\int 3 = \int 2 + \int 1 = 3 + 1 = 4$$
$$\int 4 = \int 3 + \int 2 = 4 + 3 = 7$$
$$\int 5 = \int 4 + \int 3 - \int 0 = 7 + 4 - 5 = 6$$
$$\int 6 = \int 5 + \int 4 - \int 1 = 6 + 7 - 1 = 12$$
$$\int 7 = \int 6 + \int 5 - \int 2 - \int 0 = 12 + 6 - 3 - 7 = 8$$
$$\int 8 = \int 7 + \int 6 - \int 3 - \int 1 = 8 + 12 - 4 - 1 = 15$$
$$\int 9 = \int 8 + \int 7 - \int 4 - \int 2 = 15 + 8 - 7 - 3 = 13$$
$$\int 10 = \int 9 + \int 8 - \int 5 - \int 3 = 13 + 15 - 6 - 4 = 18$$
$$\int 11 = \int 10 + \int 9 - \int 6 - \int 4 = 18 + 13 - 12 - 7 = 12$$
$$\int 12 = \int 11 + \int 10 - \int 7 - \int 5 + \int 0 = 12 + 18 - 8 - 6 + 12$$
$$\int 13 = \int 12 + \int 11 - \int 8 - \int 6 + \int 1 = 28 + 12 - 15 - 12 + 1 = 14$$
$$\int 14 = \int 13 + \int 12 - \int 9 - \int 7 + \int 2 = 14 + 28 - 13 - 8 + 3 = 24$$
$$\int 15 = \int 14 + \int 13 - \int 10 - \int 8 + \int 3 + \int 0 = 24 + 14 - 18 - 15 + 4 + 15 = 24$$
$$\int 16 = \int 15 + \int 14 - \int 11 - \int 9 + \int 4 + \int 1 = 24 + 24 - 12 - 13 + 7 + 1 = 31$$
$$\int 17 = \int 16 + \int 15 - \int 12 - \int 10 + \int 5 + \int 2 = 31 + 24 - 28 - 18 + 6 + 3 = 18$$
$$\int 18 = \int 17 + \int 16 - \int 13 - \int 11 + \int 6 + \int 3 = 18 + 31 - 14 - 12 + 12 + 4 = 39$$
$$\int 19 = \int 18 + \int 17 - \int 14 - \int 12 + \int 7 + \int 4 = 39 + 18 - 24 - 28 + 8 + 7 = 20$$
$$\int 20 = \int 19 + \int 18 - \int 15 - \int 13 + \int 8 + \int 5 = 20 + 39 - 24 - 14 + 15 + 6 = 42$$

I believe these examples sufficient to imagine that it is not by sheer luck that my rule finds itself in agreement with reality.

7. If nevertheless one objects that these examples only prove the correctness of the first six terms of our sequence: 1, 2, 5, 7, 12, 15, and not that of the law of progression which I indicated, it will suffice to choose, in order to verify this law, some examples using larger numbers.

I. Let 101 be the number for which we wish to find the sum of its divisors. We have:

$$\begin{aligned}\textstyle\int 101 &= \textstyle\int 100 + \int 99 - \int 96 - \int 94 + \int 89 + \int 86 - \int 79 - \int 75 \\ &\quad \textstyle + \int 66 + \int 61 - \int 50 - \int 44 + \int 31 + \int 24 - \int 9 - \int 1 \\ &= 217 + 156 - 252 - 144 + 90 + 132 - 80 - 124 \\ &\quad + 144 + 62 - 93 - 84 + 32 + 60 - 13 - 1\end{aligned}$$

which, adding these numbers two at a time

$$\textstyle\int 101 = 373 - 396 + 222 - 204 + 206 - 177 + 92 - 14$$

which gives $\int 101 = 102$, from which we conclude that 101 is a prime number, if we didn't know it already.

II. Let 301 be the number for which we wish to know the sum of its divisors, and we will have

$$\begin{aligned}\textstyle\int 301 = &\textstyle\int 300 + \int 299 - \int 296 - \int 294 + \int 289 + \int 286 - \int 279 - \int 275 \\ &\textstyle + \int 266 + \int 261 - \int 250 - \int 244 + \int 231 + \int 224 - \int 209 - \int 201 \\ &\textstyle + \int 184 + \int 175 - \int 156 - \int 146 + \int 125 + \int 114 - \int 91 - \int 79 \\ &\textstyle + \int 54 + \int 41 - \int 14 - \int 0\end{aligned}$$

where it is clear how, by means of the differences, we can easily form this sequence for each case proposed. Now, substituting the sums of the divisors, we will find

$$\begin{aligned}\textstyle\int 301 = &+868 - 570 + 307 - 416 + 480 - 468 + 384 - 240 + 360 - 392 \\ &+ 156 - 112 + 336 - 684 + 504 - 372 + 390 - 434 + 504 - 272 \\ &+ 248 - 222 + 240 - 80 + 120 - 24 + 42 - 301\end{aligned}$$

where

$$\textstyle\int 301 = +4939 - 4587 = 353$$

from which we recognize that 301 is not prime. Now since $301 = 7 \cdot 43$, we will have

$$\textstyle\int 301 = \int 7 \cdot \int 43 = 8 \cdot 44 = 353$$

by the rule just shown.

8. These examples that I have just developed will no doubt remove any scruple which one could still have about the truth of my formula. But one could be all the more surprised by this nice property, not seeing any connection between the composition of my formula and the nature of the divisors, the sum of which the proposition centers upon. The progression of the numbers 1, 2, 5, 7, 12, etc. appears not only to have no relation to the subject in question, but, seeing that the law of these numbers is interrupted and that they are a mixture of two different regular progressions, that is

$$1, 5, 12, 22, 35, 51, \text{etc.} \quad \text{and} \quad 2, 7, 15, 26, 40, 57, \text{etc.},$$

it almost seems that such an irregularity would not find a place in analysis. Furthermore, the lack of a proof must in no small way increase the interest in this, seeing that it would be almost morally impossible to arrive at the discovery of such a property, without having been led there by a sure method, which might be able to take the place of a perfect proof. I also admit that it is not by pure luck that I fell upon this discovery, but another proposition of a similar nature which must be judged true, though I cannot prove it, opened the way for me to arrive at this nice property. And although this investigation centers only on the nature of numbers, to which infinite analysis would not seem to have any applicability, it is nevertheless by means of differentiation and other detours that I was led to this conclusion. I would hope that one could find a shorter and more natural way to get there, and perhaps consideration of the route I followed will lead to it.

9. A long time ago I considered, on the occasion of the problem of the partition of numbers, this expression

$$(1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)(1-x^8)\cdots$$

imagining it to continue to infinity. I explicitly multiplied a large number of these factors together, to see the form of the series that would result, and I found this progression:

$$1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \cdots$$

where the exponents of x are the same numbers that enter into the earlier formula, and also the signs $+$ and $-$ alternate two at a time. One has only to undertake this multiplication and to continue it as far as one judges appropropriate, to convince oneself of the truth of this series. Indeed I have no other proof for this than a long induction, which I at least pushed far enough along so that I do not have the slightest doubt about the law by which these terms and their exponents are formed. I searched a long time in vain for a rigorous proof that this series must be equal to the proposed expression $(1-x)(1-x^2)(1-x^3)$ etc. and I put the same request to a few of my friends whom I knew to be strong in these sorts of questions. But all have fallen into agreement with me about the truth of this conversion, without having been able to unearth any source of proof. So it will be a known truth, however not yet proved, that if one puts

$$s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)\cdots$$

the same quantity s can also be expressed by

$$s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \cdots$$

because each person is in a position to convince himself of this truth by explicit calculation to such a point as he would wish; and it would seem impossible that the law which we have discovered for 20 terms, for example, would not be equally true for all the following ones.

10. Having then discovered that these two infinite expressions are equal, although the equality might not be proved, all the conclusions that we might deduce from this equality will be of the same nature, which is to say truths not proved. Alternatively, if any one of these conclusions could be proved, we could conversely derive from it a proof of the mentioned equality; and it is in this view that I worked these two expressions in several ways, and was led among other things to the discovery that I just explained, whose truth must be as certain as that of the equality of these two expressions. This is the manner in which I operated. These two expressiones being equal,

$$\begin{array}{rl} \text{I.} & s = (1-x)(1-x^2)(1-x^3)(1-x^4)(1-x^5)(1-x^6)(1-x^7)\cdots \\ \text{II.} & s = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} + \cdots \end{array}$$

in order to clear the first equation of its factors, I take logarithms, and get

$$\ell\, s = \ell(1-x) + \ell(1-x^2) + \ell(1-x^3) + \ell(1-x^4) + \ell(1-x^5) + \cdots$$

Now, to eliminate the logarithms, I take the differentials, which gives this equation

$$\frac{ds}{s} = -\frac{dx}{1-x} - \frac{2x\,dx}{1-x^2} - \frac{3x^2\,dx}{1-x^3} - \frac{4x^3\,dx}{1-x^4} - \frac{5x^4\,dx}{1-x^5} - \cdots$$

which I divide by $-dx$ and multiply by x, to get:

$$-\frac{x\,ds}{s\,dx} = \frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \cdots$$

The second value for this same quantity s gives by differentiation

$$ds = -dx - 2x\,dx + 5x^4\,dx + 7x^6\,dx - 12x^{11}\,dx - 15x^{14}\,dx + \cdots$$

from which, by multiplying by $-x$ and dividing by $s\,dx$, we derive another value for $-\frac{x\,ds}{s\,dx}$ which will be

$$-\frac{x\,ds}{s\,dx} = \frac{x + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - 22x^{22} - 26x^{26} + \cdots}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots}$$

11. Let the value of $-\frac{x\,ds}{s\,dx} = t$, and we will have two equal values for this quantity t

$$\text{I.}\quad t = \frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \frac{4x^4}{1-x^4} + \frac{5x^5}{1-x^5} + \frac{6x^6}{1-x^6} + \cdots$$

$$\text{II.}\quad t = \frac{x + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - 22x^{22} - 26x^{26} + \cdots}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots}$$

I resolve each term of the first expression into a geometric progression by ordinary division, and I get:

$$\begin{array}{lcccccccccccccc}
t = & x & + x^2 & + x^3 & + x^4 & + x^5 & + x^6 & + x^7 & + x^8 & + x^9 & + x^{10} & + x^{11} & + x^{12} & + \cdots \\
 & & +2x^2 & & +2x^4 & & +2x^6 & & +2x^8 & & +2x^{10} & & +2x^{12} & + \cdots \\
 & & & +3x^3 & & & +3x^6 & & & +3x^9 & & & +3x^{12} & + \cdots \\
 & & & & +4x^4 & & & & +4x^8 & & & & +4x^{12} & + \cdots \\
 & & & & & +5x^5 & & & & & +5x^{10} & & & + \cdots \\
 & & & & & & +6x^6 & & & & & & +6x^{12} & + \cdots \\
 & & & & & & & +7x^7 & & & & & & + \cdots \\
 & & & & & & & & +8x^8 & & & & & + \cdots \\
 & & & & & & & & & +9x^9 & & & & + \cdots \\
 & & & & & & & & & & +10x^{10} & & & + \cdots \\
 & & & & & & & & & & & +11x^{11} & & + \cdots \\
 & & & & & & & & & & & & +12x^{12} & + \cdots
\end{array}$$

where it is easy to see that each power of x occurs as many times as its exponent has divisors, since each divisor becomes a coefficient of this same power of x. Thus, gathering like powers into one sum, the coefficient of each power of x will be the sum of all the divisors of that exponent. And therefore, expressing these sums of divisors by prefixing the sign $\int$, like I did above, I will obtain for t the series which follows

$$t = \int 1 \cdot x + \int 2 \cdot x^2 + \int 3 \cdot x^3 + \int 4 \cdot x^4 + \int 5 \cdot x^5 + \int 6 \cdot x^6 + \int 7 \cdot x^7 + \cdots$$

from which the law of progression is altogether manifest; and, although it seems that induction has some part in the determination of these coefficients, when one considers the preceding infinite expression, one will be easily assured of the necessity of this law of progression.

12. Let us substitute this value in place of t in the second second expression for this same letter t, which, cleared of fractions, reduces to this form:

$$\begin{aligned} 0 = {} & t(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - \cdots) \\ & - x - 2x^2 + 5x^5 - 7x^7 - 12x^{12} - 15x^{15} + 22x^{22} + 26x^{26} + \cdots \end{aligned}$$

Now, putting the preceding value of t into this equation, we will find

$$\begin{aligned} 0 = {} & \int 1 \cdot x + \int 2 \cdot x^2 + \int 3 \cdot x^3 + \int 4 \cdot x^4 + \int 5 \cdot x^5 + \int 6 \cdot x^6 + \int 7 \cdot x^7 + \int 8 \cdot x^8 + \int 9 \cdot x^9 + \ldots \\ & - x - \int 1 \cdot x^2 - \int 2 \cdot x^3 - \int 3 \cdot x^4 - \int 4 \cdot x^5 - \int 5 \cdot x^6 - \int 6 \cdot x^7 - \int 7 \cdot x^8 - \int 8 \cdot x^9 - \ldots \\ & - 2x^2 - \int 1 \cdot x^3 - \int 2 \cdot x^4 - \int 3 \cdot x^5 - \int 4 \cdot x^6 - \int 5 \cdot x^7 - \int 6 \cdot x^8 - \int 7 \cdot x^9 - \ldots \\ & + 5x^5 + \int 1 \cdot x^6 + \int 2 \cdot x^7 + \int 3 \cdot x^8 + \int 4 \cdot x^9 + \ldots \\ & + 7x^7 + \int 1 \cdot x^8 + \int 2 \cdot x^9 + \ldots \end{aligned}$$

First, it is easy to observe that the coefficients of each power of x is the sum of divisors: first the exponent of this power itself, and then the other smaller numbers which result when we successively subtract from the exponent the numbers 1, 2, 5, 7, 12, 15, 22, 26, etc. Second, if the exponent of the power of x is equal to a term from this numerical sequence, then this same term also goes with the coefficients. Third, the order of the signs needs no explanation. Thus, we will conclude in general that the power x^n will have these coefficients:

$$\int n - \int(n-1) - \int(n-2) + \int(n-5) + \int(n-7) - \int(n-12) - \int(n-15) + \cdots$$

all the way until we get to the negative numbers. But if any of these numbers prefixed by the sign $\int$ is zero, then we must put in its place the number n itself, so that in this case, we have $\int 0 = n$ and the sign of this term follows the general order of the others.

13. So then, since the infinite expression of the preceding § must be equal to zero regardless of the value we give to the quantity x, it follows of necessity that the coefficients of each separate power, taken together, must be equal to zero, and therefore we will have the following equations:

$$\left.\begin{array}{l}
\int 1 - 1 = 0 \\
\int 2 - \int 1 - 2 = 0 \\
\int 3 - \int 2 - \int 1 = 0 \\
\int 4 - \int 3 - \int 2 = 0 \\
\int 5 - \int 4 - \int 3 + 5 = 0 \\
\int 6 - \int 5 - \int 4 + \int 1 = 0 \\
\int 7 - \int 6 - \int 5 + \int 2 + 7 = 0
\end{array}\right\} \text{ or } \left\{\begin{array}{l}
\int 1 = 1 \\
\int 2 = \int 1 + 2 \\
\int 3 = \int 2 + \int 1 \\
\int 4 = \int 3 + \int 2 \\
\int 5 = \int 4 + \int 3 - 5 \\
\int 6 = \int 5 + \int 4 - \int 1 \\
\int 7 = \int 6 + \int 5 - \int 2 - 7
\end{array}\right.$$

and in general we will have:

$$0 = \int n - \int(n-1) - \int(n-2) + \int(n-5) + \int(n-7) - \int(n-12) - \int(n-15) + \cdots$$

and consequently

$$\int n = \int(n-1) + \int(n-2) - \int(n-5) - \int(n-7) + \int(n-12) + \int(n-15) - \cdots$$

which is the same expression I gave above and which expresses the law according to which the sums of the divisors of natural numbers are continued. In addition to the reason for the signs and for the nature of the progression of the numbers

$$1,\ 2,\ 5,\ 7,\ 12,\ 15,\ 22,\ 26,\ 35,\ 40,\ 51,\ 57,\ 70,\ 77,\ \ldots$$

we also see, by what I have just put forward, the reason why, in the case where the term $\int 0$ occurs, we must put in its place the number n itself, which could have seemed the strangest part of my expression. This reasoning, although it is still very far from a perfect proof, will nevertheless permit the lifting of several doubts concerning the bizarre form of the expression which I just expanded upon.