Introduction to Analysis of the Infinite
Chapter 18: Continued Fractions

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356. Having treated at length in the preceding chapters both infinite series, and products combining infinitely many factors, it would seem appropriate to say a few words about a third kind of infinite expression, those containing continued fractions or divisions. Though this kind of expression has indeed been hardly developed thus far, we nevertheless do not doubt that it will find ample use in the analysis of the infinite. In fact I have already shown several examples of this kind, which reinforces this expectation in no small way. But in this chapter, it is principally applications to arithmetic and common algebra that I set out to briefly point out and explain.

357. By a *continued fraction*, I mean a fraction whose denominator consists of a whole number added to a fraction, which itself in turn has for a denominator a whole number and fractional part formed in the same manner as the preceding, and so on in sequence, whether there be an infinite number of such fractions, or a finite number of them.

Such are the following expressions:

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\cdots}}}}} \qquad \text{or} \qquad a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\cdots}}}}}$$

In the first, the numerators of all the fractions are unity. Those are the ones we will principally consider. In the second, the numerators are arbitrary numbers.

358. Having shown the form of these continued fractions, the next thing is to discover how they might be represented in the way that fractions are ordinarily expressed. To do this easily, let us proceed to break them off in steps, starting at the first fraction, then at the second, then at the third, and

Introductio in analysin infinitorum. Tomus primus. 1748, pp. 295–320. Number 101 in the Eneström index.

so on. Doing this clearly yields

$$a = a$$

$$a + \frac{1}{b} = \frac{ab+1}{b}$$

$$a + \cfrac{1}{b + \cfrac{1}{c}} = \frac{abc+a+c}{bc+1}$$

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d}}} = \frac{abcd+ab+ad+cd+1}{bcd+b+d}$$

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e}}}} = \frac{abcde+abe+ade+cde+abc+a+c+e}{bcde+be+de+bc+1}$$

etc.

359. Although the rule which governs how to construct the numerator and denominator out of the letters a, b, c, d, etc., is not easily discerned, nevertheless some attention will reveal that each fraction may be formed from the ones preceding it. Each numerator, in fact, is the last numerator multiplied by the new letter, plus the numerator before that; and the denominators follow the same rule. So by writing the letters a, b, c, d, etc., in a row like this

$$\begin{array}{ccccc} a & b & c & d & e \\ \dfrac{1}{0}, & \dfrac{a}{1}, & \dfrac{ab+1}{b}, & \dfrac{abc+a+c}{bc+1}, & \dfrac{abcd+ab+ad+cd+1}{bcd+b+d} \end{array}$$

it is easy to find the next fraction from the ones already found: multiply the last numerator already found by the letter above it, then add to this product the numerator before that; and the same rule applies to the denominators. In order that this rule can be applied from the start, I have prefixed the fraction $1/0$ which, although it does not originate from a continued fraction, it nevertheless makes the rule of progression more apparent. Moreover, each fraction shows the value of the continued fraction which is continued up to and including the letter written above the term which precedes it.

360. Similarly, the other form of continued fraction,

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\cdots}}}}}$$

will give, when broken off in the exact same places, the following values

$$a = a$$

$$a+\frac{\alpha}{b} = \frac{ab+\alpha}{b}$$

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c}} = \frac{abc+\beta a+\alpha c}{bc+\beta}$$

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d}}} = \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b}$$

etc.,

in which each fraction will be found from its two predecessors as follows

$$\begin{array}{ccccc} a & b & c & d & e \\ \dfrac{1}{0}, & \dfrac{a}{1}, & \dfrac{ab+\alpha}{b}, & \dfrac{abc+\beta a+\alpha c}{bc+\beta}, & \dfrac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b}. \\ \alpha & \beta & \gamma & \delta & \varepsilon \end{array}$$

361. That is to say, written above the fractions to be formed are the indices a, b, c, d, etc., and written below them, the indices α, β, γ, δ, etc. The first fraction will again be set to $1/0$, and the second to $a/1$. Then the subsequent fractions will be found as follows. The immediately preceding numerator is multiplied by the index above it, but then the numerator before that one is multiplied by the index below that one, and then the two products are added together. This sum will be the numerator of the next fraction. Similarly, its denominator will be the sum two products: the immediately preceding denominator multiplied by the index above it, and the denominator before that one multiplied by the index below that one. Each fraction found in this way will provide the value for the continued fraction which is continued up to and including the denominator indicated by the letter written above the preceding term.

362. So if one of these fractions is continued for as long as indices are available, then the final fraction will give the true value of that continued fraction. But the preceding fractions ever-more-closely approach this value, and for that reason they provide an exceedingly convenient approximation for it. Let us set x to be the true value of the continued fraction

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cdots}}}}$$

Then it is clear that the first fraction $1/0$ is greater than x, but the second $a/1$ will be less than x, and the third $a+\alpha/b$ will again be greater, the fourth once again less, and so on, where these fractions will alternate being greater, then less, than x. Furthermore, it is clear that each fraction is closer to x than any of its predecessors is, which gives us a way to quickly and conveniently approximate the value of x itself, even if the continued fraction were to go on indefinitely, provided that the numerators α, β, γ, δ, etc., do not grow excessively. However, if all those numerators are unity, then the approximation is subject to no such inconvenience.

363. In order to better understand the reason this approximation approaches the true value of the continued fraction, let us consider the differences of the above fractions. Ignoring the first $1/0$, the difference between the second and the third is

$$\frac{\alpha}{b}\,.$$

The fourth subtracted from the third yields

$$\frac{\alpha\beta}{b(bc+\beta)}\,.$$

The fourth subtracted from the fifth yields

$$\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)}\,,$$

etc. Hence the value of the continued fraction will be expressed by an ordinary series of terms as follows

$$x=a+\frac{\alpha}{b}-\frac{\alpha\beta}{b(bc+\beta)}+\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)}-\cdots\,,$$

where the series terminates when the continued fraction does not go on forever.

364. We just found a way of converting an arbitrary continued fraction into a series of terms whose signs alternate, whenever the first letter a vanishes. For example, if the fraction is

$$x = \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \cfrac{\varepsilon}{f + \cdots}}}}}$$

by the method above it will be converted to

$$x = \frac{\alpha}{b} - \frac{\alpha\beta}{b(bc+\beta)} + \frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)} - \frac{\alpha\beta\gamma\delta}{(bcd+\beta d+\gamma b)(bcde+\beta de+\gamma be+\delta bc+\beta\delta)} + \cdots,$$

from which it follows that if α, β, γ, δ, etc., are non-increasing numbers, for example all ones, but the denominators arbitrary positive integers, then the value of the continued fraction will be expressed by a quickly-converging series of terms.

365. That established, an arbitrary series of alternating terms will, in turn, be able to be converted into a continued fraction, meaning a continued fraction can be found whose value equals the sum of the given series. Let such a series be given,

$$x = A - B + C - D + E - F + \cdots,$$

and by comparing it termwise with the series arising from the continued fraction, we will have

$$A = \frac{\alpha}{b}, \qquad \text{so} \quad \alpha = Ab,$$

$$\frac{B}{A} = \frac{\beta}{bc+\beta}, \qquad \text{so} \quad \beta = \frac{Bbc}{A-B},$$

$$\frac{C}{B} = \frac{\gamma b}{bcd+\beta d+\gamma b}, \qquad \gamma = \frac{Cd(bc+\beta)}{b(B-C)},$$

$$\frac{D}{C} = \frac{\delta(bc+\beta)}{bcde+\beta de+\gamma be+\delta bc+\beta\delta}, \qquad \delta = \frac{De(bcd+\beta d+\gamma b)}{(bc+\beta)(C-D)}$$

etc.

But since

$$\beta = \frac{Bbc}{A-B}\,,$$

we will have

$$bc + \beta = \frac{Abc}{A-B}\,;$$

from which

$$\gamma = \frac{ACcd}{(A-B)(B-C)}\,.$$

Further,

$$\begin{aligned} bcd + \beta d + \gamma b &= (bc+\beta)d + \gamma b \\ &= \frac{Abcd}{A-B} + \frac{ACbcd}{(A-B)(B-C)} \\ &= \frac{ABbcd}{(A-B)(B-C)}\,, \end{aligned}$$

and so we will get

$$\frac{bcd + \beta d + \gamma b}{bc+\beta} = \frac{Bd}{B-C}$$

and

$$\delta = \frac{BDde}{(B-C)(C-D)}\,.$$

Similarly, we will find

$$\varepsilon = \frac{CEef}{(C-D)(D-E)}$$

and so on.

366. In order to clarify this rule, let us set

$$\begin{aligned} P &= b\,, \\ Q &= bc + \beta\,, \\ R &= bcd + \beta d + \gamma b\,, \\ S &= bcde + \beta de + \gamma be + \delta bc + \beta\delta\,, \\ T &= bcdef + \cdots\,, \\ V &= bcdefg + \cdots\,, \\ &\text{etc.,} \end{aligned}$$

and by the rule of these expressions, we will have

$$Q = Pc + \beta\,,$$
$$R = Qd + \gamma P\,,$$
$$S = Re + \delta Q\,,$$
$$T = Sf + \varepsilon R\,,$$
$$V = Tg + \xi S\,,$$
$$\text{etc.}$$

And so by having introduced these letters, we will obtain

$$x = \frac{\alpha}{P} - \frac{\alpha\beta}{PQ} + \frac{\alpha\beta\gamma}{QR} - \frac{\alpha\beta\gamma\delta}{RS} + \frac{\alpha\beta\gamma\delta\varepsilon}{ST} - \cdots\,.$$

367. Therefore, since we are setting

$$x = A - B + C - D + E - F + \cdots\,,$$

we will have

$$A = \frac{\alpha}{P}\,, \qquad \alpha = AP\,,$$
$$\frac{B}{A} = \frac{\beta}{Q}\,, \qquad \beta = \frac{BQ}{A}\,,$$
$$\frac{C}{B} = \frac{\gamma P}{R}\,, \qquad \gamma = \frac{CR}{BP}\,,$$
$$\frac{D}{C} = \frac{\delta Q}{S}\,, \qquad \delta = \frac{DS}{CQ}\,,$$
$$\frac{E}{D} = \frac{\varepsilon R}{T}\,, \qquad \varepsilon = \frac{ET}{DR}$$
$$\text{etc.} \qquad \text{etc.}$$

But further, taking differences will yield

$$A - B = \frac{\alpha(Q-\beta)}{PQ} = \frac{\alpha c}{Q} = \frac{APc}{Q}$$

$$B - C = \frac{\alpha\beta(R-\gamma P)}{PQR} = \frac{\alpha\beta d}{PR} = \frac{BQd}{R}$$

$$C - D = \frac{\alpha\beta\gamma(S-\delta Q)}{QRS} = \frac{\alpha\beta\gamma e}{QS} = \frac{CRe}{S}$$

$$D - E = \frac{\alpha\beta\gamma\delta(T-\varepsilon R)}{RST} = \frac{\alpha\beta\gamma\delta f}{RT} = \frac{DSf}{T}$$

etc.

Then if they are multiplied together in pairs, we will get

$$(A-B)(B-C) = ABcd \cdot \frac{P}{R} \quad \text{and} \quad \frac{R}{P} = \frac{ABcd}{(A-B)(B-C)}$$

$$(B-C)(C-D) = BCde \cdot \frac{Q}{S} \quad \text{and} \quad \frac{S}{Q} = \frac{BCed}{(B-C)(C-D)}$$

$$(C-D)(D-E) = CDef \cdot \frac{R}{T} \quad \text{and} \quad \frac{T}{R} = \frac{CDef}{(C-D)(D-E)}$$

etc.,

from which, since

$$P = b\,,$$

$$Q = \frac{\alpha c}{A-B} = \frac{Abc}{A-B},$$

we will have

$$\alpha = Ab\,,$$

$$\beta = \frac{Bbc}{A-B}\,,$$

$$\gamma = \frac{ACcd}{(A-B)(B-C)}\,,$$

$$\delta = \frac{BDde}{(B-C)(C-D)}\,,$$

$$\varepsilon = \frac{CEef}{(C-D)(D-E)}$$

etc.

368. Having found the values of the numerators α, β, γ, δ, etc., the denominators b, c, d, e, etc., are left to our discretion. It is appropriate that they be chosen not only so that they are whole numbers themselves, but also so that they produce integer values for α, β, γ, δ, etc. But this depends on the nature of the numbers A, B, C, etc., whether they are integers or fractions. Let us take them to be integers, and then the requirement will be satisfied by setting

$$\begin{array}{lcl} b = 1 & & \alpha = A \\ c = A - B & & \beta = B \\ d = B - C & \text{from which} & \gamma = AC \\ e = C - D & & \delta = BD \\ f = D - E & & \varepsilon = CE \\ \quad\text{etc.} & & \quad\text{etc.} \end{array}$$

Consequently, if we set

$$x = A - B + C - D + E - F + \cdots ,$$

then the very same value of x can be expressed as a continued fraction by

$$x = \cfrac{A}{1 + \cfrac{B}{A - B + \cfrac{AC}{B - C + \cfrac{BD}{C - D + \cfrac{CE}{D - E + \cdots}}}}}$$

369. On the other hand, if all the terms of the series are fractional numbers such that

$$x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \cdots ,$$

then the following values for α, β, γ, δ, etc., will be obtained

$$\alpha = \frac{b}{A},$$
$$\beta = \frac{Abc}{B-A},$$
$$\gamma = \frac{B^2cd}{(B-A)(C-B)},$$
$$\delta = \frac{C^2de}{(C-B)(D-C)},$$
$$\varepsilon = \frac{D^2ef}{(D-C)(E-D)}$$
etc.

Therefore let us set as follows

$$\begin{array}{lcl} b = A & \text{from which} & \alpha = 1, \\ c = B - A & & \beta = AA, \\ d = C - B & & \gamma = BB, \\ e = D - C & & \delta = CC \end{array}$$
etc.,

and we will get for the continued fraction

$$x = \cfrac{1}{A + \cfrac{AA}{B - A + \cfrac{BB}{C - B + \cfrac{CC}{D - C + \cdots}}}}$$

Example I.

Transform this infinite series

$$x = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$$

into a continued fraction.

We accordingly set $A = 1$, $B = 2$, $C = 3$, $D = 4$, etc., and because the given series has a value equal to $\log 2$, we will have

$$\log 2 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \cfrac{25}{1 + \cdots}}}}}}$$

Example II.

Transform this infinite series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

where π *denotes the circumference of a circle whose diameter equals* 1, *into a continued fraction.*

Substituting for A, B, C, D, etc., the numbers 1, 3, 5, 7, etc., yields

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cdots}}}}}$$

and so, by inverting the fraction we will get

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cdots}}}}$$

which is the expression Brouncker first advanced for the quadrature of the circle.

Example III.

If given an infinite series such as

$$x = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \cdots$$

on account of

$$A = m, \quad B = m+n, \quad C = m+2n, \quad \text{etc.},$$

it is transformed into this continued fraction

$$x = \cfrac{1}{m + \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \cdots}}}}}$$

which by inverting becomes

$$\frac{1}{x} - m = \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \cdots}}}}$$

Example IV.

Because in §178 above we found that

$$\frac{\pi \cos \frac{m\pi}{n}}{n \sin \frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \cdots$$

we will have, for forming the continued fraction

$$A = m, \quad B = n - m, \quad C = n + m, \quad D = 2n - m, \quad \cdots,$$

from which

$$\frac{\pi \cos \frac{m\pi}{n}}{n \sin \frac{m\pi}{n}} = \cfrac{1}{m + \cfrac{mm}{n - 2m + \cfrac{(n-m)^2}{2m + \cfrac{(n+m)^2}{n - 2m + \cfrac{(2n-m)^2}{2m + \cfrac{(2n+m)^2}{n - 2m + \cdots}}}}}}$$

370. If the given series progresses by successive factors, such that

$$x = \frac{1}{A} - \frac{1}{AB} + \frac{1}{ABC} - \frac{1}{ABCD} + \frac{1}{ABCDE} - \cdots,$$

then the following determinations will be put forth

$$\alpha = \frac{b}{A}\,,$$
$$\beta = \frac{bc}{B-1}\,,$$
$$\gamma = \frac{Bcd}{(B-1)(C-1)}\,,$$
$$\delta = \frac{Cde}{(C-1)(D-1)}\,,$$
$$\varepsilon = \frac{Def}{(D-1)(E-1)}$$
etc.

Accordingly, let us set the following

$$\begin{array}{lcl} b = A & \text{from which} & \alpha = 1\,, \\ c = B-1 & & \beta = A\,, \\ d = C-1 & & \gamma = B\,, \\ e = D-1 & & \delta = C \\ f = E-1 & & \varepsilon = D \end{array}$$
etc.

Then as a consequence we will have

$$x = \cfrac{1}{A + \cfrac{A}{B - 1 + \cfrac{B}{C - 1 + \cfrac{C}{D - 1 + \cfrac{D}{E - 1 + \cdots}}}}}$$

Example I.

We previously defined e to be the number whose logarithm equals 1, and found that

$$\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{1\cdot 2} - \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} - \cdots$$

or alternatively

$$1 - \frac{1}{e} = \frac{1}{1} - \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} - \frac{1}{1\cdot 2\cdot 3\cdot 4} + \cdots$$

This series will be converted into a continued fraction by setting

$$A = 1, \quad B = 2, \quad C = 3, \quad D = 4, \quad \cdots$$

and when this is done we will have

$$1 - \frac{1}{e} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \cdots}}}}}}$$

from which, by eliminating the initial asymmetry, we will get

$$\frac{1}{e-1} = \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{4}{4 + \cfrac{5}{5 + \cdots}}}}}}$$

Example II.

We previously found the cosine of any arc which equals its chosen radius to be

$$1 - \frac{1}{2} + \frac{1}{2 \cdot 12} - \frac{1}{2 \cdot 12 \cdot 30} + \frac{1}{2 \cdot 12 \cdot 30 \cdot 56} - \cdots$$

Accordingly, let us set

$$A = 1, \quad B = 2, \quad C = 12, \quad D = 30, \quad E = 56, \quad \cdots$$

and also set x to the cosine of an arc which equals its radius. Then we will have

$$x = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \cdots}}}}}$$

or alternatively

$$\frac{1}{x} - 1 = \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \cdots}}}}$$

371. If the series from above is joined with a geometric series, which is to say

$$x = A - Bz + Cz^2 - Dz^3 + Ez^4 - Fz^5 + \cdots,$$

then we will have

$$\alpha = Ab,$$

$$\beta = \frac{Bbcz}{A - Bz},$$

$$\gamma = \frac{ACcdz}{(A - Bz)(B - Cz)},$$

$$\delta = \frac{BDdez}{(B - Cz)(C - Dz)},$$

$$\varepsilon = \frac{CEefz}{(C - Dz)(D - Ez)}$$

etc.

Let us now set

$$\begin{array}{lcl} b = 1 & \text{and so} & \alpha = A, \\ c = A - Bz & & \beta = Bz, \\ d = B - Cz & & \gamma = ACz, \\ e = C - Dz & & \delta = BDz \end{array}$$

etc.,

from which

$$x = \cfrac{A}{1 + \cfrac{Bz}{A - Bz + \cfrac{ACz}{B - Cz + \cfrac{BDz}{C - Dz + \cdots}}}}$$

372. In order to allow a more general result, let us set

$$x = \frac{A}{L} - \frac{By}{Mz} + \frac{Cy^2}{Nz^2} - \frac{Dy^3}{Oz^3} + \frac{Ey^4}{Pz^4} - \cdots,$$

and then by comparing it to what has been established, we will have

$$\alpha = \frac{Ab}{L},$$

$$\beta = \frac{BLbcy}{AMz - BLy},$$

$$\gamma = \frac{ACM^2cdyz}{(AMz - BLy)(BNz - CMy)},$$

$$\delta = \frac{BDN^2deyz}{(BNz - CMy)(COz - DNy)},$$

etc.

The values of b, c, d, etc., are set as follows

$$\begin{array}{lcl} b = L & \text{so} & \alpha = A, \\ c = AMz - BLy & & \beta = BLLy, \\ d = BNz - CMy & & \gamma = ACM^2yz, \\ e = COz - DNy & & \delta = BDN^2yz, \\ f = DPz - EOy & & \varepsilon = CDO^2yz \end{array}$$

etc.,

from which the given series will be expressed by the following continued fraction

$$x = \cfrac{A}{L + \cfrac{BLLy}{AMz - BLy + \cfrac{ACMMyz}{BNz - CMy + \cfrac{BDNNyz}{COz - DNy + \cdots}}}}$$

373. Finally, let the given series be of the form

$$x = \frac{A}{L} - \frac{ABy}{LMz} + \frac{ABCy^2}{LMNz^2} - \frac{ABCDy^3}{LMNOz^3} + \cdots,$$

and the following values will be put forth

$$\alpha = \frac{Ab}{L},$$

$$\beta = \frac{Bbcy}{Mz - By},$$

$$\gamma = \frac{CMcdyz}{(Mz - By)(Nz - Cy)},$$

$$\delta = \frac{DNdeyz}{(Nz - Cy)(Oz - Dy)},$$

$$\varepsilon = \frac{EOefyz}{(Oz - Dy)(Pz - Ey)},$$

etc.

So in order to find integer values let us set

$$\begin{array}{lcl} b = Lz & \text{so} & \alpha = Az, \\ c = Mz - By & & \beta = BLyz, \\ d = Nz - Cy & & \gamma = CMyz, \\ e = Oz - Dy & & \delta = DNyz, \\ f = Pz - Ey & & \varepsilon = EOyz \end{array}$$

etc.,

from which the value of the given series will be expressed as

$$x = \cfrac{Az}{Lz + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \cdots}}}}$$

Or so that the law of progression may be made plain starting from the beginning,

$$\frac{Az}{x} - Ay = Lz - Ay + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \cdots}}}$$

374. In this way, innumerably many infinite continued fractions will be able to be found, whose true values may be produced. For, as has been treated above, an infinite series whose sum is known can be applied to this task. Each and every one of them will be able to be transformed into a continued fraction whose value is, moreover, equal to the sum of that series. The examples which have just been related suffice to show this usage. It is still to be desired to discover a method whose benefit would be, when given an arbitrary continued fraction, its value could immediately be found. Although a continued fraction can be transformed into an infinite series whose sum may be investigated by known methods, nevertheless, for the most part such series which arise are so intricate that their sums, though they may be simple enough, are scarcely able to be found.

375. So that it may be more clearly discerned that there are continued fractions of this kind, whose values may easily be determined by other means, yet nothing much may be gathered from the infinite series they are converted into, let us consider this continued fraction

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$$

whose denominators are all equal to each other. If we form fractions in the manner shown above

$$\begin{array}{cccccccc} 0 & 2 & 2 & 2 & 2 & 2 & 2 & \\ \frac{1}{0}, & \frac{0}{1}, & \frac{1}{2}, & \frac{2}{5}, & \frac{5}{12}, & \frac{12}{29}, & \frac{29}{70}, & \text{etc.,} \end{array}$$

then this series will arise

$$x = 0 + \frac{1}{2} - \frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 12} - \frac{1}{12 \cdot 29} + \frac{1}{29 \cdot 70} - \cdots,$$

or, if the terms are joined pairwise, we will have

$$x = \frac{2}{1 \cdot 5} + \frac{2}{5 \cdot 29} + \frac{2}{29 \cdot 169} + \cdots$$

or

$$x = \frac{1}{2} - \frac{2}{2 \cdot 12} - \frac{2}{12 \cdot 70} - \ldots$$

Moreover, since

$$\begin{aligned} x = \frac{1}{4} - \frac{1}{2 \cdot 2 \cdot 5} + \frac{1}{2 \cdot 5 \cdot 12} - \frac{1}{2 \cdot 12 \cdot 29} + \ldots \\ + \frac{1}{4} - \frac{1}{2 \cdot 2 \cdot 5} + \frac{1}{2 \cdot 5 \cdot 12} - \frac{1}{2 \cdot 12 \cdot 29} + \ldots, \end{aligned}$$

we will have

$$x = \frac{1}{4} + \frac{1}{1\cdot 5} - \frac{1}{2\cdot 12} + \frac{1}{5\cdot 29} - \frac{1}{12\cdot 70} + \ldots$$

which series, though strongly convergent, its true sum nevertheless cannot be gathered from its form.

376. For these kinds of continued fractions, in which the denominators are either all equal, or else they repeat, such that if several terms are truncated from the beginning, what remains is equal to the fraction itself, there is an easy way to investigate these sums. In the previous example, since

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$$

we will have

$$x = \frac{1}{2+x}\,,$$

and so

$$xx + 2x = 1\,,$$

and

$$x + 1 = \sqrt{2}\,,$$

so that the value of this continued fraction is

$$\sqrt{2} - 1\,.$$

But the fractions which come from a continued fraction before having fully extracted the root approach this value ever more closely, and so quickly that it is only with difficulty that a quicker way may be found to approximately express this irrational value by rational numbers. Indeed, $\sqrt{2}-1$ is so close to $29/70$ that the error is negligible: by extracting the root we will get

$$\sqrt{2} - 1 = 0.41421356236\,,$$

and

$$\frac{29}{70} = 0.41428571428\,,$$

so the error consists only in the 100 000th place.

377. Just as continued fractions provide a convenient method to approximate the value of $\sqrt{2}$, they will similarly provide the easiest way to approximate roots of other numbers under investigation. To this end, let us set

$$x = \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cdots}}}}}$$

and we will have

$$x = \frac{1}{a + x}$$

and

$$xx + ax = 1\,,$$

from which

$$x = -\frac{1}{2}a + \sqrt{1 + \frac{1}{4}aa} = \frac{\sqrt{aa+4} - a}{2}\,.$$

So this continued fraction will serve to approximate the value of the square root of the number $aa + 4$. Moreover, by successively substituting in place of a the numbers 1, 2, 3, 4, etc., we will obtain the numbers $\sqrt{5}$, $\sqrt{2}$, $\sqrt{13}$, $\sqrt{5}$, $\sqrt{29}$, $\sqrt{10}$, $\sqrt{53}$, etc., these roots conveyed in the simplest form, as follows

$$\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & & \\ \frac{0}{1}, & \frac{1}{1}, & \frac{1}{2}, & \frac{2}{3}, & \frac{3}{5}, & \frac{5}{8}, & \text{etc.} = & \frac{\sqrt{5}-1}{2}, \end{array}$$

$$\begin{array}{cccccccc} 2 & 2 & 2 & 2 & 2 & 2 & & \\ \frac{0}{1}, & \frac{1}{2}, & \frac{2}{5}, & \frac{5}{12}, & \frac{12}{29}, & \frac{29}{70}, & \text{etc.} = & \sqrt{2}-1\,, \end{array}$$

$$\begin{array}{cccccccc} 3 & 3 & 3 & 3 & 3 & 3 & & \\ \frac{0}{1}, & \frac{1}{3}, & \frac{3}{10}, & \frac{10}{33}, & \frac{33}{109}, & \frac{109}{360}, & \text{etc.} = & \frac{\sqrt{13}-3}{2}, \end{array}$$

$$\begin{array}{cccccccc} 4 & 4 & 4 & 4 & 4 & 4 & & \\ \frac{0}{1}, & \frac{1}{4}, & \frac{4}{17}, & \frac{17}{72}, & \frac{72}{305}, & \frac{305}{1292}, & \text{etc.} = & \sqrt{5}-2\,, \end{array}$$

etc.

It is also to be noted that the approximation goes faster as the number a gets larger. In the last example, we have

$$\sqrt{5} = 2\tfrac{305}{1292}\,,$$

where the error is less than $1/(1292 \cdot 5473)$, and 5473 is the denominator of the next fraction $1292/5473$.

378. But by this method, the roots of other numbers will not be able to be extracted, unless they are the sum of two squares. So in order to extend this approximation to all numbers, let us set

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \cdots}}}}}}$$

We will have

$$x = \cfrac{1}{a + \cfrac{1}{b + x}} = \frac{b + x}{ab + 1 + ax}\,,$$

and so

$$axx + abx = b$$

and

$$x = -\frac{1}{2}b \pm \sqrt{\frac{1}{4}bb + \frac{b}{a}} = \frac{-ab + \sqrt{aabb + 4ab}}{2a}\,.$$

We will now be able to find the roots of all numbers. For example, by setting $a = 2$ and $b = 7$ we will get

$$x = \frac{-14 + \sqrt{14 \cdot 18}}{4} = \frac{-7 + 3\sqrt{7}}{2}\,,$$

but the approximate value of x itself will be shown by the following fractions

$$\begin{array}{cccccc} 2 & 7 & 2 & 7 & 2 & 7 \\ \frac{0}{1}, & \frac{1}{2}, & \frac{7}{15}, & \frac{15}{32}, & \frac{112}{239}, & \frac{239}{510}, \end{array} \quad \text{etc.}$$

We will therefore have the approximation

$$\frac{-7 + 3\sqrt{7}}{2} = \frac{239}{510}$$

and

$$\sqrt{7} = \frac{2024}{765} = 2.645\,751\,6\,.$$

But it actually is

$$\sqrt{7} = 2.645\,751\,31\,,$$

so that the error is less than 3/10 000 000.

379. Let us go one more step, and set

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \cdots}}}}}}}$$

We will have

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + x}}} = \cfrac{1}{a + \cfrac{c + x}{bx + bc + 1}} = \frac{bx + bc + 1}{(ab + 1)x + abc + a + c}\,,$$

from which

$$(ab + 1)xx + (abc + a - b + c)x = bc + 1\,,$$

and so

$$x = \frac{-abc - a + b - c + \sqrt{(abc + a + b + c)^2 + 4}}{2(ab + 1)}\,,$$

where the quantity under the radical sign is again the sum of two squares. Therefore, this form will not serve to extract the roots of other numbers, unless the first form already sufficed. In a similar way, if the the denominators of the continued fraction consisted of the four letters a, b, c, d being continuously repeated, then it would not serve any better than the second form, which contained only two letters, and so on.

380. Just as continued fractions can be so usefully employed to extract square roots, they will likewise serve to solve quadratic equations. This is clear from the above calculation, at least when x is determined from the particular quadratic equation. But moreover, a root of any quadratic equation can be expressed this way, by a continued fraction. Given the equation

$$xx = ax + b\,,$$

since

$$x = a + \frac{b}{x}\,,$$

by substituting into the last term the value of x already found, we will get

$$x = a + \cfrac{b}{a + \cfrac{b}{x}}$$

and proceeding in the same way, we will find an infinite continued fraction

$$x = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \ldots}}}$$

which, however, because the numerators b are not equal to unity, is not so convenient to use.

381. In order to show their use in arithmetic, the first thing to note is that any ordinary fraction can be converted into a continued fraction. Let x be a fraction

$$x = \frac{A}{B}\,,$$

in which $A > B$. Let A be divided by B, the quotient be a and the remainder C. Then this remainder C divides the previous divisor B, producing quotient b and leaving remainder D. This then divides the previous divisor C, and this operation, which is widely used to compute the greatest common divisor of two numbers A and B being investigated, continues to completion, as follows

$$\begin{array}{lllll}
B\,)\,\underline{A}\,(\,a & & & & \\
C\,)\,\underline{B}\,(\,b & & & & \\
D\,)\,\underline{C}\,(\,c & & & & \\
E\,)\,\underline{D}\,(\,d & & & & \\
F\,)\ \text{etc.} & & & &
\end{array}$$

And then we will have, by the nature of division

$$A = \alpha B + C\,, \quad \text{and so} \quad \frac{A}{B} = a + \frac{C}{B}\,,$$

$$B = bC + D\,, \qquad \frac{B}{C} = b + \frac{D}{C}\,, \quad \frac{C}{B} = \frac{1}{b + \frac{D}{C}}\,,$$

$$C = cD + E\,, \qquad \frac{C}{D} = c + \frac{E}{D}\,, \quad \frac{D}{C} = \frac{1}{c + \frac{E}{D}}\,,$$

$$D = dE + F\,, \qquad \frac{D}{E} = d + \frac{F}{E}\,, \quad \frac{E}{D} = \frac{1}{d + \frac{F}{E}}\,,$$

etc.

From here, by substituting the latter values into the previous ones, we will get

$$x = \frac{A}{B} = a + \frac{C}{B} = a + \frac{1}{b + \frac{D}{C}} = a + \cfrac{1}{b + \cfrac{1}{c + \frac{E}{D}}}$$

and in the end x will be expressed as a quotient purely in terms of the a, b, c, d, etc., we found above, in the following way

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \cdots}}}}}$$

Example I.

Given the fraction 1461/59, it will be converted into a continued fraction, all of whose numerators are unity, as follows. Let us set up the very same calculation which is normally used to find the greatest common divisor of the numbers 59 and 1461.

$$
\begin{array}{l}
59 \;)\; 1461 \;(\; 24 \\
\quad\;\; \underline{118} \\
\quad\;\;\; 281 \\
\quad\;\;\; \underline{236} \\
\quad\;\;\;\; 45 \;)\; 59 \;(\; 1 \\
\qquad\qquad\; \underline{45} \\
\qquad\qquad\; 14 \;)\; 45 \;(\; 3 \\
\qquad\qquad\qquad\;\; \underline{42} \\
\qquad\qquad\qquad\;\;\; 3 \;)\; 14 \;(\; 4 \\
\qquad\qquad\qquad\qquad\;\; \underline{12} \\
\qquad\qquad\qquad\qquad\;\;\; 2 \;)\; 3 \;(\; 1 \\
\qquad\qquad\qquad\qquad\qquad\; \underline{2} \\
\qquad\qquad\qquad\qquad\qquad\; 1 \;)\; 2 \;(\; 2 \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{2} \\
\qquad\qquad\qquad\qquad\qquad\qquad 0
\end{array}
$$

From these quotients we will get

$$\frac{1461}{59} = 24 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}}}$$

Example II.

Even decimal fractions can be converted in the very same way. Let us consider

$$\sqrt{2} = 1.41421356 = \frac{141\,421\,356}{100\,000\,000},$$

from which we set up this calculation

100 000 000	141 421 356	1
82 842 712	100 000 000	2
17 157 288	41 421 356	2
14 213 560	34 314 576	2
2 943 728	7 106 780	2
2 438 648	5 887 456	2
505 080	1 219 324	2
418 728	1 010 160	2
etc.	209 364	

From this calculation, all the denominators are now seen to be 2, and moreover

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}}$$

the reason for which has already been explained above.

Example III.

But the number e, whose logarithm equals 1, is at this point worth particular attention. We have

$$e = 2.718281828459\,,$$

and so

$$\frac{e-1}{2} = 0.8591409142295\,,$$

whose decimal fraction, if handled in the above manner, will yield the following quotients

8 591 409 142 295	10 000 000 000 000	1
8 451 545 146 224	8 591 409 142 295	6
139 863 996 071	1 408 590 857 704	10
139 312 557 916	1 398 639 960 710	14
551 438 155	9 950 896 994	18
550 224 488	9 925 886 790	22
1 213 667	25 010 204	etc.

If that calculation were painstakingly continued further toward the value of e itself, then these quotients would be obtained

$$1, \quad 6, \quad 10, \quad 14, \quad 18, \quad 22, \quad 26, \quad 30, \quad 34, \quad \cdots$$

which, except for the first, sets out an arithmetic progression, from which it is clear that

$$\frac{e-1}{2} = \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \cfrac{1}{22 + \cdots}}}}}}$$

and the rationale of the fraction can be obtained from infinitesimal calculus.

382. Just as, from these kinds of expressions, fractions may be determined which readily lead to the true value of the expression, so too this method may be applied to a decimal fraction, expressing it with ordinary fractions, to be determined, which approximate it closely. Moreover, if a fraction is given whose numerator and denominator are extremely large, fractions of smaller fixed numbers can be found which, although not exactly equal to the given fraction, nevertheless differ from it minimally. From here, the problem formerly treated by Wallis can easily be solved, where it is to find fractions expressed using smaller numbers, which exhaust as much as possible the value of a given fraction expressed with larger numbers, without increasing the numbers. Fractions arising from our method so closely approach the value of the continued fraction they were derived from, that no fixed numbers can be given which approach it better, unless those numbers are larger.

Example I.

Let us express the ratio of the diameter to the circumference using numbers which are as economical as possible, such that the accuracy cannot be increased unless larger numbers are introduced. If the known decimal fraction

$$3.1415926535\cdots$$

is evolved by continued division in the manner set out previously, we will obtain the following quotients

$$3, \quad 7, \quad 15, \quad 1, \quad 292, \quad 1, \quad 1, \quad \cdots,$$

from which the following fractions will be formed

$$\frac{1}{0}, \quad \frac{3}{1}, \quad \frac{22}{7}, \quad \frac{333}{106}, \quad \frac{355}{113}, \quad \frac{103993}{33102}, \quad \cdots.$$

The second fraction now shows that the diameter being to the circumference as $1 : 3$ can certainly not be given more accurately without larger numbers. The third fraction gives the Archimedean ratio $7 : 22$, and the fifth that of Metius, which comes so close to the true value that the error is less than $1/(113 \cdot 33102)$. The rest of these fractions alternate between being larger and smaller than the true value.

Example II.

Let us express, using smallest numbers, the approximate ratio of the day to the average solar year. Since such a year is $365^d\,5^h\,48'\,55''$, the year will contain, as an ordinary fraction

$$365\,\frac{20935}{86400}$$

days. So the only thing needed is to evolve this fraction, which will give the following quotients

$$4, \quad 7, \quad 1, \quad 6, \quad 1, \quad 2, \quad 2, \quad 4$$

yielding the fractions

$$\frac{0}{1}, \quad \frac{1}{4}, \quad \frac{7}{29}, \quad \frac{8}{33}, \quad \frac{55}{227}, \quad \frac{63}{260}, \quad \frac{181}{747}, \quad \ldots$$

Therefore the hours, along with the minutes and seconds which surpass 365 days make about one day in four years, which is the origin of the Julian calendar. But more precisely, 33 years fill up 8 days, or 747 years 181 days, from which it follows that in 400 years there are 97 extra days. Hence, whereas the Julian calendar inserts 100 days during this interval, the Gregorian calendar converts three of the leap years into regular years.