erit ergo proxime $z = \frac{1681.45}{2408.60} = \frac{1681.3}{2408.4} = \frac{5043}{9632} = 0,52356$: CAP. XVII

At ex proportione Peripheriæ ad Diametrum cognita debebat esse $z = 0,523598$, ita ut radix inventa tantum parte $\frac{3}{100000}$ a vero discrepet. Hoc autem in hac æquatione commode usu venit, quod ejus omnes radices sint reales, atque a minima reliquæ satis notabiliter discrepent. Quæ conditio cum rarissime in æquationibus infinitis locum habeat, huic methodo ad eas resolvendas parum usus relinquitur.

CAPUT XVIII.

De fractionibus continuis.

356. Quoniam in præcedentibus Capitibus plura, cum de Seriebus infinitis, tum de productis ex infinitis Factoribus conflatis disserui, non incongruum fore visum est, si etiam nonnulla de tertio quodam expressionum infinitarum genere addidero, quod continuis fractionibus vel divisionibus continetur. Quanquam enim hoc genus parum adhuc est excultum, tamen non dubitamus, quin ex eo amplissimus usus in analysin infinitorum aliquando sit redundaturus. Exhibui enim jam aliquoties ejusmodi specimina, quibus hæc expectatio non parum probabilis redditur. Imprimis vero ad ipsam Arithmeticam & Algebram communem non contemnenda subsidia affert ista speculatio, quæ hoc Capite breviter indicare atque exponere constitui.

357. Fractionem autem continuam voco ejusmodi fractionem, cujus denominator constat ex numero integro cum fractione, cujus denominator denuo est aggregatum ex integro & fractione, quæ porro simili modo sit comparata, sive ista affectio in infinitum progrediatur sive alicubi sistatur. Hujusmodi ergo fractio continua erit sequens expressio

$a +$

LIB. I.

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\&c.}}}}} \quad \text{vel} \quad a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\&c.,}}}}}$$

in quarum forma priori omnes fractionum numeratores sunt unitates, quam potissimum hic contemplabor, in altera vero forma sunt numeratores numeri quicunque.

358. Exposita ergo fractionum harum continuarum forma, primum videndum est, quemadmodum earum significatio consueto more expressa inveniri queat. Quæ ut facilius inveniri possit, progrediamur per gradus, abrumpendo illas fractiones primo in prima, tum in secunda, post in tertia & ita porro fractione; quo facto patebit fore

$$a = a$$

$$a+\frac{1}{b} = \frac{ab+1}{b}$$

$$a+\cfrac{1}{b+\cfrac{1}{c}} = \frac{abc+a+c}{bc+1}$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d}}} = \frac{abcd+ab+ad+cd+1}{bcd+b+d}$$

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e}}}} = \frac{abcde+abe+ade+cde+abc+a+c+e}{bcde+be+de+bc+1}$$

&c.

359. Etsi in his fractionibus ordinariis non facile lex, secundum quam numerator ac denominator ex litteris a, b, c, d, &c., componantur, perspicitur, tamen attendenti statim patebit, quemadmodum quælibet fractio ex præcedentibus formari queat. Quilibet enim numerator est aggregatum ex numeratore ultimo per novam litteram multiplicato, & ex numeratore

meratore penultimo ſimplici: eademque lex in denominatoribus obſervatur. Scriptis ergo ordine litteris a, b, c, d, &c., ex iis fractiones inventæ facile formabuntur hoc modo CAP. XVIII.

$$\overset{a}{\frac{1}{0}};\quad \overset{b}{\frac{a}{1}};\quad \overset{c}{\frac{ab+1}{b}};\quad \overset{d}{\frac{abc+a+c}{bc+1}};\quad \overset{e}{\frac{abcd+ab+ad+cd+1}{bcd+b+d}}$$

ubi quilibet numerator invenitur, ſi præcedentium ultimus per indicem ſupra ſcriptum multiplicetur atque ad productum antepenultimus addatur; quæ eadem lex pro denominatoribus valet. Quo autem hac lege ab ipſo initio uti liceat, præfixi fractionem $\frac{1}{0}$ quæ, etiamſi e fractione continua non oriatur, tamen progreſſionis legem clariorem efficit. Quælibet autem fractio exhibet valorem fractionis continuæ uſque ad eam litteram, quæ antecedenti imminet, incluſive continuata.

360. Simili modo altera fractionum continuarum forma

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\&c.}}}}},$$

dabit, prout aliis aliiſque locis abrumpitur, ſequentes valores

$$a = a$$

$$a+\frac{\alpha}{b} = \frac{ab+\alpha}{b}$$

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c}} = \frac{abc+\beta a+\alpha c}{bc+\beta}$$

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d}}} = \frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b}$$

&c.,

 quarum

quarum fractionum quæque ex binis præcedentibus sequentem in modum invenietur

$$\overset{a}{\underset{\alpha}{\frac{1}{0}}};\ \overset{b}{\underset{\beta}{\frac{a}{1}}};\ \overset{c}{\underset{\gamma}{\frac{ab+\alpha}{b}}};\ \overset{d}{\underset{\delta}{\frac{abc+\beta a+\alpha c}{bc+\beta}}};\ \overset{e}{\underset{\varepsilon}{\frac{abcd+\beta ad+\alpha cd+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b}}}$$

361. Fractionibus scilicet formandis supra inscribantur indices a, b, c, d, &c., infra autem subscribantur indices α, β, γ, δ, &c.. Prima fractio iterum constituatur $\frac{1}{0}$, secunda $\frac{a}{1}$, tum sequentium quævis formabitur si antecedentium ultimæ numerator per indicem supra scriptum, penultimæ vero numerator per indicem infra scriptum multiplicetur & ambo producta addantur, aggregatum erit numerator fractionis sequentis: simili modo ejus denominator erit aggregatum ex ultimo denominatore per indicem supra scriptum, & ex penultimo denominatore per indicem infra scriptum multiplicatis. Quælibet vero fractio hoc modo inventa præbebit valorem fractionis continuæ ad eum usque denominatorem, qui fractioni antecedenti est inscriptus, continuatæ inclusive.

362. Quod si ergo hæ fractiones eousque continuentur quoad fractio continua indices suppeditet, tum ultima fractio verum dabit valorem fractionis continuæ. Præcedentes fractiones vero continuo propius ad hunc valorem accedent, ideoque perquam idoneam appropinquationem suggerent. Ponamus enim verum valorem fractionis continuæ

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\&c.,}}}}\quad \text{esse} = x$$

atque manifestum est fractionem primam $\frac{1}{0}$ esse majorem quam

quam x; secunda vero $\frac{a}{1}$ minor erit quam x; tertia $a + \frac{\alpha}{b}$ Cap. XVIII.
iterum vero valore erit major; quarta denuo minor, atque ita porro hæ fractiones alternatim erunt majores & minores quam x. Porro autem perspicuum est quamlibet fractionem propius accedere ad verum valorem x quam ulla præcedentium; unde hoc pacto citissime & commodissime valor ipsius x proxime obtinetur; etiamsi fractio continua in infinitum progrediatur, dummodo numeratores α, β, γ, δ, &c., non nimis crescant; sin autem omnes isti numeratores fuerint unitates, tum appropinquatio nulli incommodo est obnoxia.

363. Quo ratio hujus appropinquationis ad verum fractionis continuæ valorem melius percipiatur, consideremus fractionum inventarum differentias. Ac, prima quidem $\frac{1}{0}$ prætermissa, differentia inter secundam ac tertiam est $= \frac{\alpha}{b}$; quarta a tertia subtracta relinquit $\frac{\alpha\beta}{b(bc+\beta)}$; quarta a quinta subtracta relinquit $\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma)}$, &c.. Hinc exprimetur valor fractionis continuæ per Seriem terminorum consuetam hoc modo, ut sit

$$x = a + \frac{\alpha}{b} - \frac{\alpha\beta}{b(bc+\beta)} + \frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)} - \&c.,$$

quæ Series toties abrumpitur quoties fractio continua non in infinitum progreditur.

364. Modum ergo invenimus fractionem continuam quamcunque in Seriem terminorum, quorum signa alternantur, convertendi, si quidem prima littera a evanescat. Si enim fuerit

LIB. I.

$$x = \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{e + \cfrac{\varepsilon}{f + \&c.}}}}},$$

erit per ea, quæ modo invenimus,

$$x = \frac{\alpha}{b} - \frac{\alpha\beta}{b(bc+\beta)} + \frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)} - \frac{\alpha\beta\gamma\delta}{(bcd+\beta d+\gamma b)(bcde+\beta de+\gamma be+\delta bc+\beta\delta)} + \&c..$$

Unde, si α, β, γ, δ, &c. fuerint numeri non crescentes, uti omnes unitates, denominatores vero a, b, c, d, &c. numeri integri quicunque affirmativi, valor fractionis continuæ exprimetur per Seriem terminorum maxime convergentem.

365. His probe consideratis, poterit vicissim Series quæcunque terminorum alternantium in fractionem continuam converti, seu fractio continua inveniri cujus valor æqualis sit summæ Seriei propositæ. Sit enim proposita hæc Series

$$x = A - B + C - D + E - F + \&c.,$$

erit, singulis terminis cum Serie ex fractione continua orta comparandis

$A = \frac{\alpha}{b}$; hincque $\alpha = Ab$,

$\frac{B}{A} = \frac{\beta}{bc+\beta}$; unde fit $\beta = \frac{Bbc}{A-B}$

$\frac{C}{B} = \frac{\gamma b}{bcd+\beta d+\gamma b}$; $\gamma = \frac{Cd(bc+\beta)}{b(B-C)}$

$\frac{D}{C} = \frac{\delta(bc+\beta)}{bcde+\beta de+\gamma be+\delta bc+\beta\delta}$; $\delta = \frac{De(bcd+\beta d+\gamma b)}{(bc+\beta)(C-D)}$

&c..

At, cum sit $\beta = \frac{Bbc}{A-B}$, erit $bc+\beta = \frac{Abc}{A-B}$; unde

$\gamma =$

$\gamma = \frac{ACcd}{(A-B)(B-C)}$. Porro fit $bcd + \beta d + \gamma b =$ CAP. XVIII.
$(bc+\beta)d+\gamma b = \frac{Abcd}{A-B} + \frac{ACbcd}{(A-B)(B-C)} = \frac{ABbcd}{(A-B)(B-C)}$,
unde erit $\frac{bcd + \beta d + \gamma b}{bc + \beta} = \frac{Bd}{B-C}$ & $\delta = \frac{BDde}{(B-C)(C-D)}$;
fimili modo reperietur $\varepsilon = \frac{CEef}{(C-D)(D-E)}$ & ita porro.

366. Quo ifta lex clarius appareat, ponamus effe

$$P = b$$
$$Q = bc + \beta$$
$$R = bcd + \beta d + \gamma b$$
$$S = bcde + \beta de + \gamma be + \delta bc + \beta\delta$$
$$T = bcdef + \&c.$$
$$V = bcdefg + \&c.,$$

erit ex lege harum expreffionum

$$Q = Pc + \beta$$
$$R = Qd + \gamma P$$
$$S = Re + \delta Q$$
$$T = Sf + \varepsilon R$$
$$V = Tg + \zeta S$$
&c..

Cum igitur his adhibendis litteris fit

$$x = \frac{\alpha}{P} - \frac{\alpha\beta}{PQ} + \frac{\alpha\beta\gamma}{QR} - \frac{\alpha\beta\gamma\delta}{RS} + \frac{\alpha\beta\gamma\delta\varepsilon}{ST} - \&c.,$$

367. Quoniam ergo ponimus effe

$$x = A - B + C - D + E - F + \&c.,$$

erit

$$A = \frac{\alpha}{P};\ \alpha = AP$$

$$\frac{B}{A} = \frac{\beta}{Q}; \quad \beta = \frac{BQ}{A}$$
$$\frac{C}{B} = \frac{\gamma P}{R}; \quad \gamma = \frac{CR}{BP}$$
$$\frac{D}{C} = \frac{\delta Q}{S}; \quad \delta = \frac{DS}{CQ}$$
$$\frac{E}{D} = \frac{\varepsilon R}{T}; \quad \varepsilon = \frac{ET}{DR}$$
&c. &c.

Porro vero differentiis ſumendis habebitur

$$A - B = \frac{\alpha(Q - \beta)}{PQ} = \frac{\alpha c}{Q} = \frac{APc}{Q}$$
$$B - C = \frac{\alpha\beta(R - \gamma P)}{PQR} = \frac{\alpha\beta d}{PR} = \frac{BQd}{R}$$
$$C - D = \frac{\alpha\beta\gamma(S - \delta Q)}{QRS} = \frac{\alpha\beta\gamma e}{QS} = \frac{CRe}{S}$$
$$D - E = \frac{\alpha\beta\gamma\delta(T - \varepsilon R)}{RST} = \frac{\alpha\beta\gamma\delta f}{RT} = \frac{DSf}{T},$$
&c. &c. &c.

Si bini igitur in ſe invicem ducantur, fiet

$$(A - B)(B - C) = ABcd.\frac{P}{R}; \quad \& \quad \frac{R}{P} = \frac{ABcd}{(A-B)(B-C)}$$
$$(B - C)(C - D) = BCde.\frac{Q}{S}; \quad \& \quad \frac{S}{Q} = \frac{BCed}{(B-C)(C-D)}$$
$$(C - D)(D - E) = CDef.\frac{R}{T}; \quad \& \quad \frac{T}{R} = \frac{CDef}{(C-D)(D-E)}$$
&c..

Unde, cum ſit $P = b$; $Q = \frac{\alpha c}{A - B} = \frac{Abc}{A - B}$, erit

$$\alpha = Ab$$
$$\beta = \frac{Bbc}{A - B}$$
$$\gamma = \frac{ACcd}{(A - B)(B - C)}$$

$\delta =$

$$\delta = \frac{BDde}{(B-C)(C-D)}$$
$$\varepsilon = \frac{CEef}{(C-D)(D-E)}$$
&c..

368. Inventis ergo valoribus numeratorum $\alpha, \beta, \gamma, \delta$, &c., denominatores b, c, d, e, &c., arbitrio nostro relinquuntur: ita autem eos assumi convenit, ut, cum ipsi sint numeri integri, tum valores integros pro $\alpha, \beta, \gamma, \delta$, &c., exhibeant. Hoc vero pendet quoque a natura numerorum A, B, C, &c., utrum sint integri an fracti. Ponamus esse numeros integros, atque quæsito satisfiet statuendo

$b = 1$		$\alpha = A$
$c = A - B$		$\beta = B$
$d = B - C$	unde fit	$\gamma = AC$
$e = C - D$		$\delta = BD$
$f = D - E$		$\varepsilon = CE$
&c.		&c.

Quocirca, si fuerit,

$$x = A - B + C - D + E - F + \text{\&c.},$$

idem ipsius x valor per fractionem continuam ita exprimi poterit, ut sit

$$x = \cfrac{A}{1 + \cfrac{B}{A - B + \cfrac{AC}{B - C + \cfrac{BD}{C - D + \cfrac{CE}{D - E + \text{\&c.}}}}}}$$

369. Sin autem omnes termini Seriei sint numeri fracti, ita ut fuerit

$$x = \frac{1}{A} - \frac{1}{B} + \frac{1}{C} - \frac{1}{D} + \frac{1}{E} - \text{\&c.},$$

habebuntur pro $\alpha, \beta, \gamma, \delta$, &c., sequentes valores

$\alpha =$

$$\alpha = \frac{b}{A}; \quad \beta = \frac{Abc}{B - A}; \quad \gamma = \frac{B^2 cd}{(B - A)(C - B)};$$

$$\delta = \frac{C^2 de}{(C - B)(D - C)}; \quad \varepsilon = \frac{D^2 ef}{(D - C)(E - D)}; \text{ \&c.}.$$

Ponatur ergo ut ſequitur

$$\begin{array}{ll} b = A; & \alpha = 1 \\ c = B - A; & \beta = AA \\ d = C - B; & \gamma = BB \\ e = D - C; & \delta = CC \end{array}$$

&c.,

eritque per fractionem continuam

$$x = \cfrac{1}{A + \cfrac{AA}{B - A + \cfrac{BB}{C - B + \cfrac{CC}{D - C + \text{\&c.}}}}},$$

EXEMPLUM I.

Transformetur hæc Series infinita

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \text{\&c.},$$

in fractionem continuam.

Erit ergo $A = 1$, $B = 2$, $C = 3$, $D = 4$, &c., atque, cum Seriei propoſitæ valor ſit $= l\,2$, erit

$$l\,2 = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \cfrac{25}{1 + \text{\&c.}}}}}}}$$

EXEMPLUM II.

Transformetur hæc Series infinita

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.,$$ CAP. XVIII.

ubi π *denotat peripheriam circuli, cujus diameter* $= 1$, *in fractionem continuam.*

Subſtitutis loco A, B, C, D, &c., numeris 1, 3, 5, 7, &c., orietur

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \&c.,}}}}}$$

hincque, invertendo fractionem, erit

$$\frac{4}{\pi} = 1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \&c.,}}}}$$

quæ eſt expreſſio, quam BROUNCKERUS primum pro quadratura circuli protulit.

EXEMPLUM III.

Sit propoſita iſta Series infinita

$$x = \frac{1}{m} - \frac{1}{m+n} + \frac{1}{m+2n} - \frac{1}{m+3n} + \&c.,$$

quæ, ob $A = m$; $B = m + n$; $C = m + 2n$; &c., in hanc fractionem continuam mutatur

$$x = \cfrac{1}{m + \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \&c.}}}}}$$

ex qua fit, invertendo,

LIB. I. $\frac{1}{x} - m = \cfrac{mm}{n + \cfrac{(m+n)^2}{n + \cfrac{(m+2n)^2}{n + \cfrac{(m+3n)^2}{n + \&c.}}}}$

EXEMPLUM IV.

Quoniam, ſupra §. 178., invenimus eſſe

$$\frac{\pi \cos. \frac{m\pi}{n}}{n \sin. \frac{m\pi}{n}} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \&c.,$$

erit, pro fractione continuanda, $A = m$; $B = n - m$; $C = n + m$; $D = 2n - m$; &c., unde fiet

$$\frac{\pi \cos. \frac{m\pi}{n}}{n \sin. \frac{m\pi}{n}} = \cfrac{1}{m + \cfrac{mm}{n-2m + \cfrac{(n-m)^2}{2m + \cfrac{(n+m)^2}{n-2m + \cfrac{(2n-m)^2}{2m + \cfrac{(2n+m)^2}{n-2m + \&c.}}}}}}$$

370. Si Series propoſita per continuos Factores progrediatur, ut ſit

$$x = \frac{1}{A} - \frac{1}{AB} + \frac{1}{ABC} - \frac{1}{ABCD} + \frac{1}{ABCDE} - \&c.,$$

tum prodibunt ſequentes determinationes

$$\alpha = \frac{b}{A};\ \beta = \frac{bc}{B-1};\ \gamma = \frac{Bcd}{(B-1)(C-1)};$$

$$\delta = \frac{Cde}{(C-1)(D-1)};\ \varepsilon = \frac{Def}{(D-1)(E-1)};\ \&c.,$$

fiat ergo ut ſequitur,

$b = A$;

$b = A$; $\qquad \alpha = 1$

$c = B - 1$; $\qquad \beta = A$

$d = C - 1$; $\qquad \gamma = B$

$e = D - 1$; $\qquad \delta = C$

$f = E - 1$; $\qquad \varepsilon = D$

&c.,

unde consequenter fiet

$$x = \cfrac{1}{A + \cfrac{A}{B - 1 + \cfrac{B}{C - 1 + \cfrac{C}{D - 1 + \cfrac{D}{E - 1 + \text{\&c.}}}}}}$$

EXEMPLUM I.

Quoniam, posito e numero cujus Logarithmus est $= 1$, supra invenimus esse

$$\frac{1}{e} = 1 - \frac{1}{1} + \frac{1}{1.2} - \frac{1}{1.2.3} + \frac{1}{1.2.3.4} - \text{\&c.},$$

seu

$$1 - \frac{1}{e} = \frac{1}{1} - \frac{1}{1.2} + \frac{1}{1.2.3} - \frac{1}{1.2.3.4} + \text{\&c.},$$

hæc Series in fractionem continuam convertetur ponendo $A = 1$, $B = 2$, $C = 3$, $D = 4$, &c.: quo ergo facto habebitur

$$1 - \frac{1}{e} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \text{\&c.},}}}}}}$$

unde, asymmetria initio rejecta, erit

$$\frac{1}{e - 1} = \cfrac{1}{1 + \cfrac{2}{2 + \cfrac{3}{3 + \cfrac{4}{4 + \cfrac{5}{5 + \text{\&c.}}}}}}$$

EXEMPLUM II.

Invenimus quoque arcus, qui radio æqualis ſumitur, coſinum eſſe $= 1 - \frac{1}{2} + \frac{1}{2.12} - \frac{1}{2.12.30} + \frac{1}{2.12.30.56} -$ &c.. Si ergo fiat $A = 1$, $B = 2$, $C = 12$, $D = 30$, $E = 56$, &c., atque Coſinus arcus qui radio æquatur, ponatur $= x$; erit

$$x = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \&c.}}}}}$$

ſeu

$$\frac{1}{x} - 1 = \cfrac{1}{1 + \cfrac{2}{11 + \cfrac{12}{29 + \cfrac{30}{55 + \&c.}}}}$$

371. Sit Series inſuper cum geometrica conjuncta, ſcilicet

$$x = A - Bz + Cz^2 - Dz^3 + Ez^4 - Fz^5 + \&c.,$$

erit

$$\alpha = Ab;\ \beta = \frac{Bbcz}{A - Bz};\ \gamma = \frac{ACcdz}{(A - Bz)(B - Cz)};$$
$$\delta = \frac{BDdez}{(B - Cz)(C - Dz)};\ \varepsilon = \frac{CEefz}{(C - Dz)(D - Ez)};\ \&c..$$

Ponatur nunc

$$b = 1; \qquad \alpha = A$$
$$c = A - Bz; \qquad \beta = Bz$$
$$d = B - Cz; \qquad \gamma = ACz$$
$$e = C - Dz; \qquad \delta = BDz,$$

unde fiet

$x =$

$$x = \cfrac{A}{1 + \cfrac{Bz}{A - Bz + \cfrac{ACz}{B - Cz + \cfrac{BDz}{C - Dz + \&c.}}}}$$

372. Quo autem hoc negotium generalius abſolvamus, ponamus eſſe

$$x = \frac{A}{L} - \frac{By}{Mz} + \frac{Cy^2}{Nz^2} + \frac{Dy^3}{Oz^3} - \frac{Ey^4}{Pz^4} + \&c.,$$

fietque, comparatione inſtituta,

$$\alpha = \frac{Ab}{L};\ \beta = \frac{BLbcy}{AMz - BLy};\ \gamma = \frac{ACM^2cdyz}{(AMz - BLy)(BNz - CMy)};$$
$$\delta = \frac{BDN^2deyz}{(BNz - CMy)(COz - DNy)};\ \&c.,$$

ſtatuantur valores b, c, d, &c., ſequenti modo

$b = L;$	erit	$\alpha = A$
$c = AMz - BLy;$		$\beta = BLLy$
$d = BNz - CMy;$		$\gamma = ACM^2yz$
$e = COz - DNy;$		$\delta = BDN^2yz$
$f = DPz - EOy;$		$\varepsilon = CEO^2yz$
&c.		&c.

unde Series propoſita per ſequentem fractionem continuam exprimetur

$$x = \cfrac{A}{L + \cfrac{BLLy}{AMz - BLy + \cfrac{ACMMyz}{BNz - CMy + \cfrac{BDNNyz}{COz - DNy + \&c.}}}}$$

373. Habeat denique Series propoſita hujuſmodi formam

$$x = \frac{A}{L} - \frac{ABy}{LMz} + \frac{ABCy^2}{LMNz^2} - \frac{ABCDy^3}{LMNOz^3} + \&c.,$$

atque ſequentes valores prodibunt

$$\alpha = \frac{Ab}{L};\ \beta = \frac{Bbcy}{Mz - By};\ \gamma = \frac{CMcdyz}{(Mz - By)(Nz - Cy)};$$
$$\delta = \frac{DNdeyz}{(Nz - Cy)(Oz - Dy)};\ \varepsilon = \frac{EOefyz}{(Oz - Dy)(Pz - Ey)};$$
&c.,

ad valores ergo integros inveniendos fiat

$b = Lz;$	erit	$\alpha = Az$
$c = Mz - By;$		$\beta = BLyz$
$d = Nz - Cy;$		$\gamma = CMyz$
$e = Oz - Dy;$		$\delta = DNyz$
$f = Pz - Ey;$		$\varepsilon = EOyz$
&c.		&c.

Unde valor Seriei propositæ ita exprimetur, ut sit

$$x = \cfrac{Az}{Lz + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \&c.}}}}.$$

Vel, ut lex progressionis statim a principio fiat manifesta, erit

$$\frac{Az}{x} - Ay = Lz - Ay + \cfrac{BLyz}{Mz - By + \cfrac{CMyz}{Nz - Cy + \cfrac{DNyz}{Oz - Dy + \&c.}}}$$

374. Hoc modo innumerabiles inveniri poterunt fractiones continuæ in infinitum progredientes, quarum valor verus exhiberi queat. Cum enim, ex supra traditis, infinitæ Series, quarum summæ constent, ad hoc negotium accommodari queant, unaquæque transformari poterit in fractionem continuam, cujus adeo valor summæ illius Seriei est æqualis. Exempla, quæ jam hic sunt allata, sufficiunt ad hunc usum ostendendum: verumtamen optandum esset, ut methodus detegeretur, cujus beneficio, si proposita fuerit fractio continua quæcunque, ejus valor immediate inveniri posset. Quanquam enim fractio continua

tinua transmutari potest in Seriem infinitam, cujus summa per methodos cognitas investigari queat, tamen plerumque istæ Series tantopere fiunt intricatæ, ut earum summa, etiamsi sit satis simplex, vix ac ne vix quidem obtineri possit. CAP. XVIII.

375. Quo autem clarius perspiciatur, dari ejusmodi fractiones continuas, quarum valor aliunde facile assignari queat, etiamsi ex Seriebus infinitis, in quas convertuntur, nihil admodum colligere liceat, consideremus hanc fractionem continuam

$$x = \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \&c.}}}},$$

cujus omnes denominatores sunt inter se æquales; si enim hinc modo supra exposito, fractiones formemus

$$\begin{array}{cccccccc} 0, & 2, & 2, & 2, & 2, & 2, & 2 & \\ \frac{1}{0}, & \frac{0}{1}, & \frac{1}{2}, & \frac{2}{5}, & \frac{5}{12}, & \frac{12}{29}, & \frac{29}{70}, & \&c.: \end{array}$$

Hinc autem porro oritur hæc Series

$$x = 0 + \frac{1}{2} - \frac{1}{2.5} + \frac{1}{5.12} - \frac{1}{12.29} + \frac{1}{29.70} - \&c.,$$

vel, si bini termini conjungantur, erit

$$x = \frac{2}{1.5} + \frac{2}{5.29} + \frac{2}{29.169} + \&c.,$$

vel

$$x = \frac{1}{2} - \frac{2}{2.12} - \frac{2}{12.70} - \&c..$$

Quin etiam, cum sit

$$x = \frac{1}{4} - \frac{1}{2.2.5} + \frac{1}{2.5.12} - \frac{1}{2.12.29} + \&c.$$

+

$$+\frac{1}{4}-\frac{1}{2.2.5}+\frac{1}{2.5.12}-\frac{1}{2.12.29}+ \&c.,$$

erit

$$x=\frac{1}{4}+\frac{1}{1.5}-\frac{1}{2.12}+\frac{1}{5.29}-\frac{1}{12.70}\ \&c.,$$

quæ Series etiamſi vehementer convergant, tamen vera earum ſumma ex earum forma colligi nequit.

376. Pro hujuſmodi autem fractionibus continuis, in quibus denominatores omnes vel ſunt æquales, vel iidem revertuntur; ita ut ea fractio, ſi ab initio aliquot terminis truncetur, toti adhuc ſit æqualis, facilis habetur modus earum ſummas explorandi. In exemplo enim propoſito, cum ſit

$$x=\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\&c.}}}},$$

erit $x=\frac{1}{2+x}$, ideoque $xx+2x=1$ & $x+1=\sqrt{2}$; ita ut valor hujus fractionis continuæ ſit $=\sqrt{2}-1$. Fractiones vero ex fractione continua ante erutæ, continuo propius ad hunc valorem accedunt, idque tam cito, ut vix promptior modus ad valorem hunc irrationalem per numeros rationales proxime exprimendum, inveniri queat. Eſt enim $\sqrt{2}-1$ tam prope $=\frac{29}{70}$, ut error ſit inſenſibilis: namque, radicem extrahendo, erit

$$\sqrt{2}-1=0,4142135623 6,$$

atque

$$\frac{29}{70}=0,4142857142 8,$$

ita ut error tantum in partibus centeſimis milleſimis conſiſtat.

377. Quemadmodum ergo fractiones continuæ commodiſſimum ſuppeditant modum ad valorem $\sqrt{2}$ appropinquandi, ita indidem

indidem facillima via aperitur ad radices aliorum numerorum proxime investigandas. Ponamus hunc in finem CAP. XVIII.

$$x = \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \cfrac{1}{a + \&c.}}}}},$$

erit $x = \frac{1}{a + x}$ & $xx + ax = 1$, unde fit $x = -\frac{1}{2}a + \sqrt{(1 + \frac{1}{4}aa)} = \frac{\sqrt{(aa + 4)} - a}{2}$. Hæc ergo fractio continua inserviet valori radicis quadratæ ex numero $aa + 4$ inveniendo. Hincque adeo substituendo loco a successive numeros 1, 2, 3, 4, &c., reperientur $\sqrt{5}$; $\sqrt{2}$; $\sqrt{13}$; $\sqrt{5}$; $\sqrt{29}$; $\sqrt{10}$; $\sqrt{53}$; &c., perductis scilicet his radicibus ad formam simplicissimam: erit ergo

$$\begin{array}{l} 1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 1, \\ \frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \&c. = \frac{\sqrt{5} - 1}{2} \\ 2,\quad 2,\quad 2,\quad 2,\quad 2,\quad 2 \\ \frac{0}{1}, \frac{1}{2}, \frac{2}{5}, \frac{5}{12}, \frac{12}{29}, \frac{29}{70}, \&c. = \sqrt{2} - 1 \\ 3,\quad 3,\quad 3,\quad 3,\quad 3,\quad 3 \\ \frac{0}{1}, \frac{1}{3}, \frac{3}{10}, \frac{10}{33}, \frac{33}{109}, \frac{109}{360}, \&c. = \frac{\sqrt{13} - 3}{2} \\ 4,\quad 4,\quad 4,\quad 4,\quad 4,\quad 4 \\ \frac{0}{1}, \frac{1}{4}, \frac{4}{17}, \frac{17}{72}, \frac{72}{305}, \frac{305}{1292}, \&c. = \sqrt{5} - 2 \\ \&c., \end{array}$$

notandum autem eo promptiorem esse approximationem, quo major fuerit numerus a: sic in ultimo exemplo erit $\sqrt{5} = 2\frac{305}{1292}$, ut error minor sit quam $\frac{1}{1292 \cdot 5473}$, ubi 5473 est denominator sequentis fractionis $\frac{1292}{5473}$.

LIB. I.

378. Hoc vero modo aliorum numerorum radices exhiberi nequeunt, nisi qui sint summa duorum quadratorum. Ut igitur hæc approximatio ad alios numeros extendatur, ponamus esse

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{a + \cfrac{1}{b + \&c.}}}}}},$$

erit

$$x = \cfrac{1}{a + \cfrac{1}{b + x}} = \frac{b + x}{ab + 1 + ax}; \text{ ideoque } axx + abx = b:$$

&

$$x = -\frac{1}{2}b \pm \surd\left(\frac{1}{4}bb + \frac{b}{a}\right) = \frac{-ab + \surd(aabb + 4ab)}{2a}.$$

Unde jam omnium numerorum radices inveniri poterunt. Sit, verbi gratia, $a = 2$, $b = 7$; erit $x = \frac{-14 + \surd 14.18}{4} = \frac{-7 + 3\surd 7}{2}$; at valorem ipsius x proxime exhibebunt sequentes fractiones

$$\begin{matrix} 2, & 7, & 2, & 7, & 2, & 7 \\ \frac{0}{1}, & \frac{1}{2}, & \frac{7}{15}, & \frac{15}{32}, & \frac{112}{239}, & \frac{239}{510}, & \&c., \end{matrix}$$

Erit ergo proxime $\frac{-7 + 3\surd 7}{2} = \frac{239}{510}$ & $\surd 7 = \frac{2024}{765} = 2,6457516$; at revera est $\surd 7 = 2,6457513 1$; ita ut error minor sit quam $\frac{3}{10000000}$.

379. Progrediamur autem ulterius ponendo

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{a + \&c.}}}}}}},$$

erit

$$x = \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + x}}} = \cfrac{1}{a + \cfrac{c + x}{bx + bc + 1}} = \frac{bx + bc + 1}{(ab + 1)x + abc + a + c};$$

unde $(ab + 1)xx + (abc + a - b + c)x = bc + 1$ atque

$x =$

$$x = \frac{-abc - a + b - c + \sqrt{((abc + a + b + c)^2 + 4)}}{2(ab + 1)};$$ Cap. XVIII.

ubi quantitas post signum radicale posita iterum est summa duorum quadratorum, neque ergo hæc forma radicibus ex aliis numeris extrahendis inservit, nisi ad quos prima forma jam suffecerat. Simili modo si quatuor litteræ a, b, c, d, continuo repetitæ denominatores fractionis continuæ constituant, tum ea plus non inserviet quam secunda, quæ duas tantum litteras continebat, & ita porro.

380. Cum igitur fractiones continuæ tam utiliter ad extractionem radicis quadratæ adhiberi queant, simul inservient æquationibus quadratis resolvendis; quod quidem ex ipso calculo est manifestum, dum x per æquationem quadraticam affectam determinatur. Potest autem vicissim facile cujusque æquationis quadratæ radix per fractionem continuam hoc modo exprimi. Sit proposita ista æquatio

$$xx = ax + b,$$

ex qua, cum sit $x = a + \frac{b}{x}$, substituatur in ultimo termino loco x valor idem jam inventus, eritque

$$x = a + \cfrac{b}{a + \cfrac{b}{x}},$$

simili ergo modo procedendo, erit per fractionem continuam infinitam

$$x = a + \cfrac{b}{a + \cfrac{b}{a + \cfrac{b}{a + \&c.}}},$$

quæ autem, cum numeratores b non sint unitates, non tam commode adhiberi potest.

381. Ut autem usus in arithmetica ostendatur, primum notandum est omnem fractionem ordinariam in fractionem con-

 tinuam

Lib. I. tinuam converti poſſe. Sit enim propoſita fractio $x = \frac{A}{B}$; in qua ſit $A > B$; dividatur A per B, ſitque quotus $= a$ & reſiduum C; tum per hoc reſiduum C dividatur præcedens diviſor B, prodeatque quotus b & relinquatur reſiduum D, per quod denuo præcedens diviſor C dividatur; ſicque hæc operatio, quæ vulgo ad maximum communem diviſorem numerorum A & B inveſtigandum uſurpari ſolet, continuetur, donec ipſa finiatur; ſequenti modo

$B)\,A\,(a$
$C)\,B\,(b$
$D)\,C\,(c$
$E)\,D\,(d$
F &c.,

eritque per naturam diviſionis

$$A = aB + C; \text{ unde } \frac{A}{B} = a + \frac{C}{B};$$

$$B = bC + D; \qquad \frac{B}{C} = b + \frac{D}{C}; \quad \frac{C}{B} = \cfrac{1}{b + \cfrac{D}{C}}$$

$$C = cD + E; \qquad \frac{C}{D} = c + \frac{E}{D}; \quad \frac{D}{C} = \cfrac{1}{c + \cfrac{E}{D}}$$

$$D = dE + F; \qquad \frac{D}{E} = d + \frac{F}{E}; \quad \frac{E}{D} = \cfrac{1}{d + \cfrac{F}{E}}$$

&c. &c. &c.

hinc, ſequentes valores in præcedentibus ſubſtituendo, erit

$$x = \frac{A}{B} = a + \frac{C}{B} = a + \cfrac{1}{b + \cfrac{D}{C}} = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{E}{D}}},$$

unde tandem x per meros quotos inventos a, b, c, d, &c., ſequentem in modum exprimetur, ut ſit

$x =$

CAP. XVIII.

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \cfrac{1}{f + \&c.}}}}}$$

EXEMPLUM I.

Sit propoſita iſta fractio $\frac{1461}{59}$, quæ ſequenti modo in fractionem continuam tranſmutabitur, cujus omnes numeratores erunt unitates. Inſtituatur ſcilicet eadem operatio, qua maximus communis diviſor numerorum 59 & 1461 quæri ſolet.

```
59) 1461 (24
    118
    ---
     281
     236
     ---
      45) 59 (1
          45
          --
          14) 45 (3
              42
              --
               3) 14 (4
                  12
                  --
                   2) 3 (1
                      2
                      -
                      1) 2 (2
                         2
                         -
                         0
```

Hinc ergo ex quotis fiet

$$\frac{1461}{59} = 24 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{2}}}}}.$$

EXEMPLUM II.

Fractiones quoque decimales eodem modo tranſmutari poterunt; ſit enim propoſita

$$\sqrt{2} = 1,41421356 = \frac{141421356}{100000000},$$

unde hæc operatio inſtituatur

100000000	141421356	1
82842712	100000000	2
17157288	41421356	2
14213560	34314576	2
2943728	7106780	2
2438648	5887456	2
505080	1219324	2
418728	1010160	2
&c.	209364	

Ex qua operatione jam patet omnes denominatores esse 2, atque adeo esse $\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\&c.}}}}}$,
cujus expressionis ratio jam ex superioribus patet.

EXEMPLUM III.

Imprimis vero etiam hic attentione dignus est numerus e, cujus logarithmus est $= 1$, qui est $e = 2,718281828459$, unde oritur $\frac{e - 1}{2} = 0,8591409142295$, quæ fractio decimalis, si superiori modo tractetur, dabit quotos sequentes

8591409142295	10000000000000	1
8451545146224	8591409142295	6
139863996071	1408590857704	10
139312557916	1398639960710	14
551438155	9950896994	18
550224488	9925886790	22
1213667	25010204	
	&c.	

si iste calculus exactius adhuc, assumto valore ipsius e, ulterius continuetur, tum prodibunt isti quoti

$$1,\ 6,\ 10,\ 14,\ 18,\ 22,\ 26,\ 30,\ 34,\ \&c.,$$

qui, demto primo, progressionem arithmeticam constituunt, unde patet fore $e - 1$

$$\frac{e-1}{2} = \cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cfrac{1}{\&c.}}}}}}},$$

cujus fractionis ratio ex calculo infinitesimali dari potest.

382. Cum igitur ex hujusmodi expressionibus fractiones erui queant, quæ quam citissime ad verum valorem expressionis deducant, hæc methodus adhiberi poterit ad fractiones decimales per ordinarias fractiones, quæ ad ipsas proxime accedant, exprimendas. Quin etiam, si fractio fuerit proposita cujus numerator & denominator sint numeri valde magni, fractiones ex minoribus numeris constantes inveniri poterunt quæ, etiamsi propositæ non sint penitus æquales, tamen ab ea quam minime discrepent. Hincque problema a WALLISIO olim tractatum facile resolvi potest, quo quæruntur fractiones minoribus numeris expressæ, quæ tam prope exhauriant valorem fractionis cujuspiam in numeris majoribus propositæ, quantum fieri poterit numeris non majoribus. Fractiones autem nostra hac methodo ortæ tam prope ad valorem fractionis continuæ, ex qua eliciuntur, accedunt, ut nullæ numeris non majoribus constantes dentur quæ propius accedant.

EXEMPLUM I.

Exprimatur ratio diametri ad peripheriam numeris tam exiguis, ut accuratior exhiberi nequeat, nisi numeri majores adhibeantur. Si fractio decimalis cognita

$$3,1415926535 \text{ \&c.},$$

modo exposito per divisionem continuam evolvatur, reperientur sequentes quoti

$$3, \quad 7, \quad 15, \quad 1, \quad 292, \quad 1, \quad 1, \quad \text{\&c.},$$

ex quibus sequentes fractiones formabuntur,

$$\frac{1}{0}, \ \frac{3}{1}, \ \frac{22}{7}, \ \frac{333}{106}, \ \frac{355}{113}, \ \frac{103993}{33102}, \ \text{\&c.},$$

secunda fractio jam ostendit esse diametrum ad peripheriam ut
1 : 3,

LIB. I. 1 : 3, neque certe numeris non majoribus accuratius dari poterit. Tertia fractio dat rationem *Archimedeam* 7 : 22, at quinta *Metianam* quæ ad verum tam prope accedit, ut error minor ſit parte $\frac{1}{113 . 33102}$. Ceterum hæ fractiones alternatim vero ſunt majores minoreſque.

EXEMPLUM II.

Exprimatur ratio diei ad annum ſolarem medium in numeris minimis proxime. Cum annus iſte ſit 365^d, 5^h, $48'$, $55''$, continebit in fractione annus unus $365 \frac{20935}{86400}$ dies. Tantum ergo opus eſt ut hæc fractio evolvatur, quæ dabit ſequentes quotos

4, 7, 1, 6, 1, 2, 2, 4

unde iſtæ eliciuntur fractiones

$$\frac{0}{1}, \frac{1}{4}, \frac{7}{29}, \frac{8}{33}, \frac{55}{227}, \frac{63}{260}, \frac{181}{747}, \&c..$$

Horæ ergo cum minutis primis & ſecundis, quæ ſupra 365 dies adſunt, quatuor annis unum diem circiter faciunt, unde calendarium *Julianum* originem habet. Exactius autem 33 annis 8 dies implentur, vel 747 annis 181 dies; unde ſequitur quadringentis annis abundare 97 dies. Quare, cum hoc intervallo calendarium *Julianum* inſerat 100 dies, *Gregorianus* quaternis ſeculis tres annos biſſextiles in communes convertit.

FINIS TOMI PRIMI.